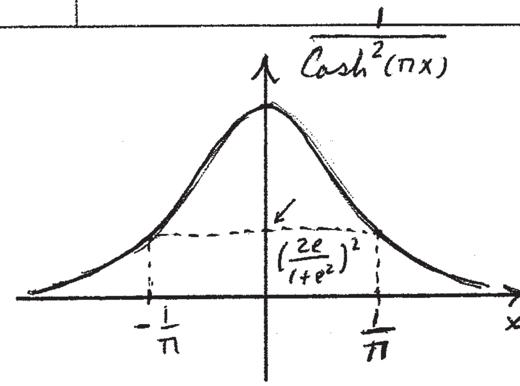
Problem 10) This problem is very similar to the previous one (Problem 9), except for the pole at $z_0 = i/2$, which is a 2nd order pole.



$$\cosh^2(\pi z_0) = \left(\frac{e^{i\pi/2}+e^{-i\pi/2}}{2}\right)^2 = \left(\frac{i-i}{2}\right)^2 = 0$$

$$\frac{d}{dz}\cosh^2(\pi z)\Big|_{z=z_0} = 2\pi \sinh(\pi z)\cosh(\pi z)\Big|_{z=i/2} = 0$$

$$\frac{d^2}{dz^2}\cosh^2(\pi z)\Big|_{z=z_0} = 2\pi^2\cosh^2(\pi z) + 2\pi^2\sinh^2(\pi z)\Big|_{z=i/2} = 0 + 2\pi^2\left(\frac{e^{i\pi/2}-e^{-i\pi/2}}{2}\right)^2 = -2\pi^2$$

Therefore, in the vicinity of the pole at $z_0 = i/2$, we may write:

$$\cosh^2(\pi z) = \cosh^2(\pi z_0) + (\cosh^2 \pi z_0)'(z-z_0) + (\cosh^2 \pi z_0)''\frac{(z-z_0)^2}{2!} + \cdots$$
$$\simeq -\pi^2(z-z_0)^2.$$

Now, $\mathcal{F}\left\{\frac{1}{\cosh^2(\pi x)}\right\} = \int_{-\infty}^{\infty}\frac{1}{\cosh^2(\pi x)}e^{-i2\pi s x}dx$ may be evaluated on the same contour as in Problem 9. The residue at $z_0 = i/2$ of the integrand is the same as the residue of $\frac{e^{-i2\pi s z}}{-\pi^2(z-z_0)^2}$. Since the pole is 2nd-order, we evaluate the derivative of $\frac{e^{-i2\pi s z}}{-\pi^2}$ at z_0, which is, $\frac{-i2\pi s e^{-i2\pi s(i/2)}}{-\pi^2}$ $= \frac{i2s}{\pi}e^{\pi s}$. Multiplication with $2\pi i$ then yields $-4se^{\pi s}$.

On the upper leg of the rectangular contour, we have $z = x+i$ and, therefore,

$$\frac{e^{-i2\pi s z}}{\cosh^2(\pi z)} = \frac{e^{2\pi s}e^{-i2\pi s x}}{\cosh^2[\pi(x+i)]} = \frac{e^{2\pi s}e^{-i2\pi s x}}{\cosh^2(\pi x)}.$$ We then write:

$$\int_{-\infty}^{\infty}\frac{e^{-i2\pi s x}}{\cosh^2 \pi x}dx - e^{2\pi s}\int_{-\infty}^{\infty}\frac{e^{-i2\pi s x}}{\cosh^2 \pi x}dx = -4se^{\pi s} \Rightarrow \int_{-\infty}^{\infty}\frac{e^{-i2\pi s x}}{\cosh^2 \pi x}dx = \frac{-4se^{\pi s}}{1-e^{+2\pi s}}$$

$$\Rightarrow \mathcal{F}\left\{\frac{1}{\cosh^2 \pi x}\right\} = \frac{-4s}{e^{-\pi s}-e^{\pi s}} = \frac{2s}{\sinh(\pi s)}.$$

Note: The function whose derivative needs to be evaluated is $\frac{(z-i/2)^2 e^{-i2\pi s z}}{\cosh^2(\pi z)}\Big|_{z=i/2}$. This requires more attention to detail, but the final result is the same.