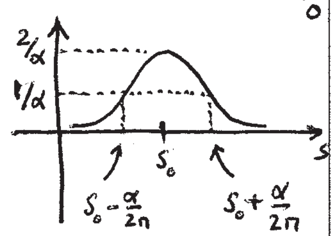
Problem 8) a) The function $\exp(i2\pi s_0 x)$ does not decay as $x \to \pm\infty$. Therefore, we multiply it with an appropriately decaying function, such as $\exp(-\alpha|x|)$, where α is a real-valued and positive constant. After finding the Fourier Transform of $e^{-\alpha|x|} e^{i2\pi s_0 x}$, we allow $\alpha \to 0$ and, in the limit, obtain the F.T. of the desired function, $e^{i2\pi s_0 x}$.

$$\mathcal{F}\left\{e^{-\alpha|x|} e^{i2\pi s_0 x}\right\} = \int_{-\infty}^{0} e^{\alpha x} e^{i2\pi s_0 x} e^{-i2\pi s x}\,dx + \int_{0}^{\infty} e^{-\alpha x} e^{i2\pi s_0 x} e^{-i2\pi s x}\,dx$$

$$= \frac{1}{\alpha - i2\pi(s-s_0)} e^{[\alpha - i2\pi(s-s_0)]x}\Big|_{-\infty}^{0} + \frac{1}{-\alpha - i2\pi(s-s_0)} e^{-[\alpha + i2\pi(s-s_0)]x}\Big|_{0}^{\infty}$$

$$= \frac{1}{\alpha - i2\pi(s-s_0)} + \frac{1}{\alpha + i2\pi(s-s_0)} = \frac{2\alpha}{\alpha^2 + 4\pi^2(s-s_0)^2}$$



The above function is symmetric around $s = s_0$, is narrow and tall when $\alpha \to 0$, and has unit area because

$$\int_{-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + 4\pi^2(s-s_0)^2}\,ds = \int_{-\pi/2}^{\pi/2} \frac{2\alpha}{\alpha^2 + \alpha^2\tan^2\theta}\,\frac{\alpha(1+\tan^2\theta)}{2\pi}\,d\theta = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} d\theta = 1.$$

Change of variable: $2\pi(s-s_0) = \alpha\tan\theta$

Therefore, in the limit when $\alpha \to 0$, the above F.T. approaches $\delta(s-s_0)$.

b) $\mathcal{F}\{\cos(2\pi s_0 x)\} = \mathcal{F}\left\{\frac{1}{2}e^{i2\pi s_0 x} + \frac{1}{2}e^{-i2\pi s_0 x}\right\} = \frac{1}{2}\delta(s-s_0) + \frac{1}{2}\delta(s+s_0)$.

$\mathcal{F}\{\sin(2\pi s_0 x)\} = \frac{1}{2i}\delta(s-s_0) - \frac{1}{2i}\delta(s+s_0)$.

$\mathcal{F}\{\cos^2(2\pi s_0 x)\} = \mathcal{F}\left\{\frac{1+\cos(4\pi s_0 x)}{2}\right\} = \frac{1}{2}\delta(s) + \frac{1}{4}\delta(s-2s_0) + \frac{1}{4}\delta(s+2s_0)$.

$\mathcal{F}\{\sin^2(2\pi s_0 x)\} = \mathcal{F}\left\{\frac{1-\cos(4\pi s_0 x)}{2}\right\} = \frac{1}{2}\delta(s) - \frac{1}{4}\delta(s-2s_0) - \frac{1}{4}\delta(s+2s_0)$.