Problem 7) a) $F(s)=\int_{-\infty}^{\infty} f(x)e^{-i2\pi sx}\,dx \Rightarrow F^*(s)=\int_{-\infty}^{\infty} f^*(x)e^{+i2\pi sx}\,dx$

But $f(x)$ is real, meaning that $f^*(x)=f(x)$. Next we change the variable from x to $y=-x$ and obtain

$$F^*(s)=\int_{-\infty}^{\infty} f(-y)e^{-i2\pi sy}\,dy=\int_{-\infty}^{\infty} f(y)e^{-i2\pi sy}\,dy=F(s)\Rightarrow F(s) \text{ is real.}$$

To prove that $F(s)$ is even we write $F(-s)=\int_{-\infty}^{\infty} f(x)e^{+i2\pi sx}\,dx=$

$$\int_{-\infty}^{\infty} f(-y)e^{-i2\pi sy}\,dy=\int_{-\infty}^{\infty} f(y)e^{-i2\pi sy}\,dy=F(s)\Rightarrow F(s) \text{ is even.}$$

b) $$G^*(s)=\int_{-\infty}^{\infty} g^*(x)e^{+i2\pi sx}\,dx=\int_{-\infty}^{\infty} g(x)e^{+i2\pi sx}\,dx=\int_{-\infty}^{\infty} g(-y)e^{-i2\pi sy}\,dy$$

$g^*(x)=g(x)$ because $g(x)$ is real

$$=-\int_{-\infty}^{\infty} g(y)e^{-i2\pi sy}\,dy=-G(s)\Rightarrow$$ Since $G^*(s)=-G(s)$, we conclude that $G(s)$ is imaginary.

$$G(-s)=\int_{-\infty}^{\infty} g(x)e^{+i2\pi sx}\,dx=\int_{-\infty}^{\infty} g(-y)e^{-i2\pi sy}\,dy=-\int_{-\infty}^{\infty} g(y)e^{-i2\pi sy}\,dy=-G(s)$$

$\Rightarrow$ $G(s)$ is odd.

c) $h_e(x)=\frac{1}{2}[h(x)+h(-x)] \Rightarrow h_e(-x)=\frac{1}{2}[h(-x)+h(x)]=h_e(x)\Rightarrow h_e(x)$ is even.

$h_o(x)=\frac{1}{2}[h(x)-h(-x)] \Rightarrow h_o(-x)=\frac{1}{2}[h(-x)-h(x)]=-h_o(x)\Rightarrow h_o(x)$ is odd.

Since $h_e(x)$ is real and even, its F.T. $H_e(s)$ must also be real and even, according to part (a). Similarly, since $h_o(x)$ is real and odd, its F.T. $H_o(s)$ must be purely imaginary and also odd, according to part (b). However, $h(x)=h_e(x)+h_o(x) \Rightarrow H(s)=H_e(s)+H_o(s)$. Clearly, the real part of $H(s)$, i.e., $H_R(s)$ is equal to $H_e(s)$, and its imaginary part, $iH_I(s)$ is equal to $H_o(s)$.