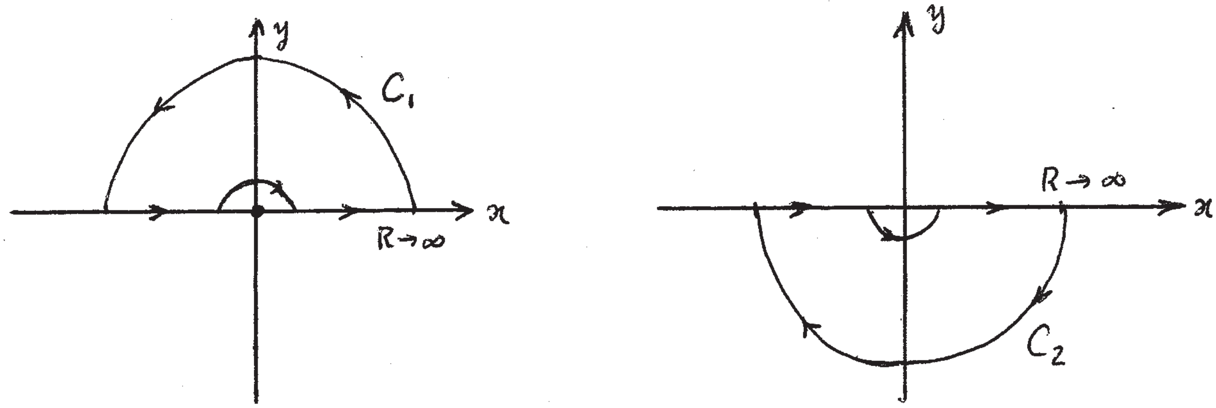
Problem 6)

$$\mathcal{F}\{\operatorname{sinc}(x)\} = \int_{-\infty}^{\infty} \frac{\sin(\pi x)}{\pi x} e^{-i2\pi s x}\,dx = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{i\pi x} - e^{-i\pi x}}{\pi x} e^{-i2\pi s x}\,dx$$

$$= \frac{1}{2\pi i}\left\{ \int_{-\infty}^{\infty} \frac{e^{-i2\pi(s-\frac{1}{2})x}}{x}\,dx - \int_{-\infty}^{\infty} \frac{e^{-i2\pi(s+\frac{1}{2})x}}{x}\,dx \right\}$$



If $s > \frac{1}{2}$ both integrals must be done on C_2, so that the Jordan lemma may be satisfied. If $s < -\frac{1}{2}$ both integrals must be taken on C_1. However, when $-\frac{1}{2} < s < +\frac{1}{2}$, the first integral must be done on C_1, and the second integral on C_2. In all these cases, the contribution of the large semi-circle is zero (Jordan's lemma). The desired integral will then be given by the value of the integral on the small semi-circle, albeit with the opposite sign. Note that the integrands have no poles inside the C_1 and C_2 contours; therefore, there are no residues in either case.

a) Small circle on C_1: $z = \epsilon e^{i\theta}$, $dz = i\epsilon e^{i\theta} d\theta$, $\displaystyle\int_{\theta=0}^{\pi} \frac{e^{-i2\pi(s\pm\frac{1}{2})\epsilon e^{i\theta}}}{\cancel{\epsilon e^{i\theta}}} i\cancel{\epsilon e^{i\theta}} d\theta$

$\xrightarrow[\epsilon\to 0]{} i\displaystyle\int_0^{\pi} d\theta = \pi i$. Note that this is the value of the integral taken counterclockwise.

b) Small circle on C_2: $z = \epsilon e^{i\theta}$, $dz = i\epsilon e^{i\theta} d\theta$, $\displaystyle\int_{\theta=\pi}^{2\pi} \frac{e^{-i2\pi(s\pm\frac{1}{2})\epsilon e^{i\theta}}}{\cancel{\epsilon e^{i\theta}}} i\cancel{\epsilon e^{i\theta}} d\theta$

$\xrightarrow[\epsilon\to 0]{} i\displaystyle\int_{\pi}^{2\pi} d\theta = \pi i$. This integral is also taken counterclockwise.

Now, when $s < -\frac{1}{2}$, both integrals are taken on C_1. Therefore,

$$\int_{-\infty}^{\infty} \frac{e^{-i2\pi(s-\frac{1}{2})x}}{x}\,dx = \int_{-\infty}^{\infty} \frac{e^{-i2\pi(s+\frac{1}{2})x}}{x}\,dx = \pi i$$

Consequently, $\mathcal{F}\{\text{sinc}(x)\} = \frac{1}{2\pi i}(\pi i - \pi i) = 0; \quad (s < -\frac{1}{2})$.

When $s > \frac{1}{2}$, both integrals are taken on C_2. Therefore,

$$\int_{-\infty}^{\infty} \frac{e^{-i2\pi(s-\frac{1}{2})x}}{x}\,dx = \int_{-\infty}^{\infty} \frac{e^{-i2\pi(s+\frac{1}{2})x}}{x}\,dx = -\pi i$$

Consequently, $\mathcal{F}\{\text{sinc}(x)\} = \frac{1}{2\pi i}(-\pi i + \pi i) = 0; \quad (s > 1/2)$.

When $-\frac{1}{2} < s < \frac{1}{2}$, the first integral is taken on C_1, yielding πi. The second integral will be taken on C_2, yielding $-\pi i$. Then

$$\mathcal{F}\{\text{sinc}(x)\} = \frac{1}{2\pi i}(\pi i + \pi i) = 1; \quad |s| < 1/2.$$

It is thus seen that $\mathcal{F}\{\text{sinc}(x)\}$ is the unit rectangular pulse, $\text{Rect}(s)$; where $\text{Rect}(s) = 0$ when $|s| > 1/2$ and 1 when $|s| < 1/2$.