Problem 5)

a) $$f(x) * \delta(x-x_0) = \int_{-\infty}^{\infty} f(x-x')\,\delta(x'-x_0)\,dx' =$$

Change of variable: $y = x' - x_0$

$$\int_{-\infty}^{\infty} f(x-x_0-y)\,\delta(y)\,dy = f(x-x_0).$$

Sifting property of $\delta(y)$

b) $$\mathcal{F}\{f(x) * \delta(x-x_0)\} = F(s)\,\mathcal{F}\{\delta(x-x_0)\} = F(s)\int_{-\infty}^{\infty}\delta(x-x_0)\,e^{-i2\pi s x}\,dx$$

$$= F(s)\int_{-\infty}^{\infty}\delta(y)\,e^{-i2\pi s(x_0+y)}\,dy = e^{-i2\pi s x_0}F(s).$$

Sifting property of $\delta(y)$.

Inverse Fourier transforming the above identity, we'll find:

$$f(x) * \delta(x-x_0) = \mathcal{F}^{-1}\{e^{-i2\pi x_0 s}F(s)\} = \int_{-\infty}^{\infty} e^{-i2\pi x_0 s}F(s)\,e^{+i2\pi s x}\,ds$$

$$= \int_{-\infty}^{\infty} F(s)\,e^{+i2\pi s(x-x_0)}\,ds = f(x-x_0).$$