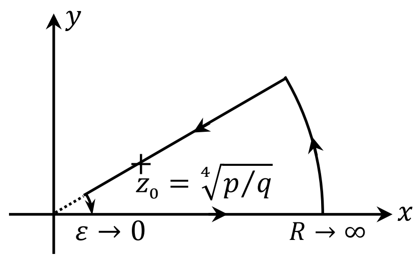
Problem 41) A change of variable from x to $y = x^{½}$ converts the original integral to

$$2\int_0^\infty \exp(-py^2 - q/y^2)\,\mathrm{d}y. \tag{1}$$

Rearranging the terms in the exponent of the above integrand now yields

$$\int_0^\infty x^{-½}\exp(-px - q/x)\,\mathrm{d}x = 2\exp(-2\sqrt{pq})\int_0^\infty \exp\left\{-\sqrt{pq}\left[(p/q)^{¼}y - \frac{1}{(p/q)^{¼}y}\right]^2\right\}\mathrm{d}y. \tag{2}$$

In the complex z-plane, the integral of $\exp[-\sqrt{pq}(z - z^{-1})^2]$ along the semi-infinite straight-line extending from $z = 0$ to ∞ in the direction of $z_0 = (p/q)^{¼}$ can be equated with the integral along the positive real axis, as shown in the figure below. This is because the integrand is analytic within the closed contour, and also because the large circular arc at infinity as well as the infinitesimal circular arc near the origin do not contribute to the loop integral.



We thus have

$$\int_0^\infty x^{-½}\exp(-px - q/x)\,\mathrm{d}x = 2\exp(-2\sqrt{pq})\,(q/p)^{¼}\int_0^\infty \exp[-\sqrt{pq}(x - x^{-1})^2]\,\mathrm{d}x$$

$$= 2\exp(-2\sqrt{pq})\,(q/p)^{¼}\left\{\int_0^1 \exp[-\sqrt{pq}(x - x^{-1})^2]\,\mathrm{d}x + \int_1^\infty \exp[-\sqrt{pq}(x - x^{-1})^2]\,\mathrm{d}x\right\}$$

$$= 2\exp(-2\sqrt{pq})\,(q/p)^{¼}\left\{\int_0^1 \exp[-\sqrt{pq}(x - x^{-1})^2]\,\mathrm{d}x + \int_0^1 y^{-2}\exp[-\sqrt{pq}(y^{-1} - y)^2]\,\mathrm{d}y\right\}$$

$$= 2\exp(-2\sqrt{pq})\,(q/p)^{¼}\int_0^1 (1 + x^{-2})\exp[-\sqrt{pq}(x - x^{-1})^2]\,\mathrm{d}x. \tag{3}$$

A change of variable from x to $\tan(\theta/2)$ now yields

$$\int_0^\infty x^{-½}\exp(-px - q/x)\,\mathrm{d}x = \exp(-2\sqrt{pq})\,(q/p)^{¼}\int_0^{\pi/2} \frac{\exp\{-\sqrt{pq}\,[\tan(\theta/2) - \cot(\theta/2)]^2\}}{\sin^2(\theta/2)\cos^2(\theta/2)}\mathrm{d}\theta$$

$$= 4\exp(-2\sqrt{pq})\,(q/p)^{¼}\int_0^{\pi/2} \frac{\exp(-4\sqrt{pq}\cot^2\theta)}{\sin^2\theta}\mathrm{d}\theta$$

$$= 4\exp(-2\sqrt{pq})\,(q/p)^{¼}\int_0^\infty \exp(-4\sqrt{pq}x^2)\,\mathrm{d}x$$

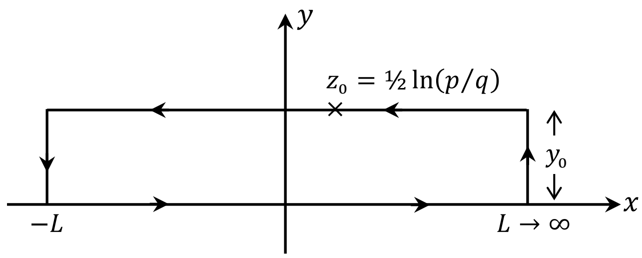
$$= \exp(-2\sqrt{pq})\,(q/p)^{¼}(pq)^{-¼}\int_{-\infty}^\infty \exp(-x^2)\,\mathrm{d}x$$

$$= \sqrt{\pi/p}\exp(-2\sqrt{pq}). \tag{4}$$

Digression: As an alternative approach to evaluating the integral, consider changing the integration variable in Eq.(1) from y to $x = 2\ln y$. We find

$$\int_0^\infty x^{-½}\exp(-px - q/x)\,\mathrm{d}x = \int_{-\infty}^\infty \exp(½x - pe^x - qe^{-x})\,\mathrm{d}x. \tag{5}$$

When x is replaced by the complex variable z, the integrand on the right-hand side of Eq.(5) will be analytic throughout the z-plane. With reference to the figure below, the integral may then be evaluated on any straight-line parallel to the real axis, namely, $z = z_0 + x$, provided that the contributions of the contour's vertical legs, where y goes from 0 to y_0 at $x = \pm L$ (with $L \to \infty$), are properly taken into account.



Equation (5) is thus simplified by defining the complex constant $z_0 = ½ \ln(p/q)$, and writing

$$p = \sqrt{pq} \times \sqrt{p/q} = \sqrt{pq}\exp[½\ln(p/q)] = \sqrt{pq}\exp(z_0), \tag{6a}$$

$$q = \sqrt{pq} \times \sqrt{q/p} = \sqrt{pq}\exp[-½\ln(p/q)] = \sqrt{pq}\exp(-z_0). \tag{6b}$$

Consequently,

$$\int_0^\infty x^{-½}\exp(-px - q/x)\,\mathrm{d}x = \int_{-\infty}^{\infty}\exp\left[½x - \sqrt{pq}\left(e^{z_0+x} + e^{-z_0-x}\right)\right]\mathrm{d}x$$

For $y_0 = \mathrm{Imag}(z_0) = ½(\varphi_p - \varphi_q)$, the contributions of vertical legs go to zero. →

$$= e^{-½z_0}\int_{\text{parallel line}}\exp\left[½z - \sqrt{pq}\left(e^{z} + e^{-z}\right)\right]\mathrm{d}z$$

$$= \sqrt[4]{q/p}\int_{\text{parallel line}}\exp\left(½z - 2\sqrt{pq}\cosh z\right)\mathrm{d}z. \tag{7}$$

The parallel line may now be returned to the real axis, and the variable z replaced with the real-valued x without any other change. Subsequently, Eq.(7) can be evaluated as follows:

$$\int_0^\infty x^{-½}\exp(-px - q/x)\,\mathrm{d}x = \sqrt[4]{q/p}\left[\int_{-\infty}^{0}\exp(½x - 2\sqrt{pq}\cosh x)\,\mathrm{d}x + \int_0^\infty \exp(½x - 2\sqrt{pq}\cosh x)\,\mathrm{d}x\right]$$

$$= 2\sqrt[4]{q/p}\int_0^\infty \exp(-2\sqrt{pq}\cosh x)\cosh(½x)\,\mathrm{d}x$$

G&R 3.547-4 →

$$= 2\sqrt[4]{q/p}\,K_{½}\left(2\sqrt{pq}\right)$$

G&R 8.469-3 →

$$= 2\sqrt[4]{q/p} \times \sqrt{\frac{\pi}{4\sqrt{pq}}}\exp(-2\sqrt{pq}) = \sqrt{\pi/p}\exp(-2\sqrt{pq}). \tag{8}$$

This, of course, is the same result as that given in Eq.(4).

Note: One may be tempted to evaluate the integral in Eq.(7) on a rectangular contour in the z-plane, by noting that adding or subtracting $2\pi i$ to z will flip the overall sign of the integral. The contributions of the vertical legs located at $x = \pm L$ (with y going from 0 to 2π, and $L \to \infty$), however, need to be properly determined as they do *not* vanish in this case. Unfortunately, this approach does *not* lead to any straightforward evaluation of the integral.