Problem 40) The real and imaginary parts of the function $h(z)$ are determined as follows:

$$h(z) = f(z)g(z) = (p + \mathrm{i}q)(r + \mathrm{i}s) = (pr - qs) + \mathrm{i}(ps + qr). \tag{1}$$

Therefore, for the product function $f(z)g(z)$, we may write

$$u(x, y) = pr - qs, \tag{2a}$$

$$v(x, y) = ps + qr. \tag{2b}$$

Confirming the Cauchy-Riemann conditions requires calculating various partial derivatives.

$$\frac{\partial u}{\partial x} = p\frac{\partial r}{\partial x} + r\frac{\partial p}{\partial x} - q\frac{\partial s}{\partial x} - s\frac{\partial q}{\partial x}, \tag{3a}$$

$$\frac{\partial u}{\partial y} = p\frac{\partial r}{\partial y} + r\frac{\partial p}{\partial y} - q\frac{\partial s}{\partial y} - s\frac{\partial q}{\partial y}, \tag{3b}$$

$$\frac{\partial v}{\partial x} = p\frac{\partial s}{\partial x} + s\frac{\partial p}{\partial x} + q\frac{\partial r}{\partial x} + r\frac{\partial q}{\partial x}, \tag{4a}$$

$$\frac{\partial v}{\partial y} = p\frac{\partial s}{\partial y} + s\frac{\partial p}{\partial y} + q\frac{\partial r}{\partial y} + r\frac{\partial q}{\partial y}. \tag{4b}$$

Since $f(z)$ is known to be differentiable at $z = z_0$, we have

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}, \tag{5a}$$

$$\frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x}. \tag{5b}$$

Similarly, since $g(z)$ is differentiable at $z = z_0$, we have

$$\frac{\partial r}{\partial x} = \frac{\partial s}{\partial y}, \tag{6a}$$

$$\frac{\partial r}{\partial y} = -\frac{\partial s}{\partial x}. \tag{6b}$$

Substituting from the above equations into Eqs.(3) now yields

$$\frac{\partial u}{\partial x} = p\frac{\partial s}{\partial y} + r\frac{\partial q}{\partial y} + q\frac{\partial r}{\partial y} + s\frac{\partial p}{\partial y}, \tag{7a}$$

$$\frac{\partial u}{\partial y} = -p\frac{\partial s}{\partial x} - r\frac{\partial q}{\partial x} - q\frac{\partial r}{\partial x} - s\frac{\partial p}{\partial x}. \tag{7b}$$

Comparing Eqs.(4) with Eqs.(7), one can readily see that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, thus confirming the satisfaction of the Cauchy-Riemann conditions for the product function $h(z)$.
