Problem 39) a) $f(z) = \exp(-z^2) = \exp[-(x+\mathrm{i}y)^2] = \exp(-x^2+y^2-2\mathrm{i}xy)$

$$= \exp(y^2-x^2)\cos(2xy) - \mathrm{i}\exp(y^2-x^2)\sin(2xy).$$

The real and imaginary parts of $f(z)$ are thus seen to be $u(x,y) = \exp(y^2-x^2)\cos(2xy)$ and $v(x,y) = -\exp(y^2-x^2)\sin(2xy)$. The partial derivatives with respect to x and y of $u(x,y)$ and $v(x,y)$ are readily found to be

$$\partial u/\partial x = -2x\exp(y^2-x^2)\cos(2xy) - 2y\exp(y^2-x^2)\sin(2xy),$$

$$\partial u/\partial y = 2y\exp(y^2-x^2)\cos(2xy) - 2x\exp(y^2-x^2)\sin(2xy),$$

$$\partial v/\partial x = 2x\exp(y^2-x^2)\sin(2xy) - 2y\exp(y^2-x^2)\cos(2xy),$$

$$\partial v/\partial y = -2y\exp(y^2-x^2)\sin(2xy) - 2x\exp(y^2-x^2)\cos(2xy).$$

Clearly, $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$. Since these Cauchy-Riemann conditions are satisfied everywhere in the complex z-plane, the function $f(z) = \exp(-z^2)$ is analytic everywhere.

b) The derivative with respect to z of $f(z)$ is given by

$$f'(z) = \partial_x u + \mathrm{i}\partial_x v = -2x\exp(y^2-x^2)\cos(2xy) - 2y\exp(y^2-x^2)\sin(2xy)$$
$$+\mathrm{i}[2x\exp(y^2-x^2)\sin(2xy) - 2y\exp(y^2-x^2)\cos(2xy)]$$
$$= -2(x+\mathrm{i}y)\exp(y^2-x^2)\cos(2xy) + \mathrm{i}2(x+\mathrm{i}y)\exp(y^2-x^2)\sin(2xy)$$
$$= -2(x+\mathrm{i}y)\exp(y^2-x^2)[\cos(2xy)+\mathrm{i}\sin(2xy)]$$

← Euler identity: $\cos\alpha + \mathrm{i}\sin\alpha = e^{\mathrm{i}\alpha}$

$$= -2z\exp(y^2-x^2)\exp(\mathrm{i}2xy) = -2z\exp[-(x^2-y^2-2\mathrm{i}xy)]$$
$$= -2z\exp[-(x+\mathrm{i}y)^2] = -2z\exp(-z^2).$$

Alternatively, one may compute the derivative of $f(z)$ by starting with the definition of the derivative, and invoking the defining property of the exponential function, i.e., $e^z = \sum_{n=0}^{\infty} z^n/n!$. One will have

$$f'(z)|_{z=z_0} = lim_{\Delta z\to 0}\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = lim_{\Delta z\to 0}\frac{\exp[-(z_0+\Delta z)^2] - \exp(-z_0^2)}{\Delta z}$$

$$= \exp(-z_0^2)\, lim_{\Delta z\to 0}\frac{\exp[-(\Delta z)^2 - 2z_0\Delta z] - 1}{\Delta z}$$

$$= \exp(-z_0^2)\, lim_{\Delta z\to 0}\frac{\cancel{1} - [(\Delta z)^2+2z_0\Delta z] + ½[(\Delta z)^2+2z_0\Delta z]^2 + \cdots \cancel{-1}}{\Delta z}$$

$$= \exp(-z_0^2)\, lim_{\Delta z\to 0}\{-(\Delta z + 2z_0) + [½(\Delta z)^2 + 2z_0\Delta z + 2z_0^2]\Delta z + \cdots\}$$

$$= -2z_0\exp(-z_0^2).$$
