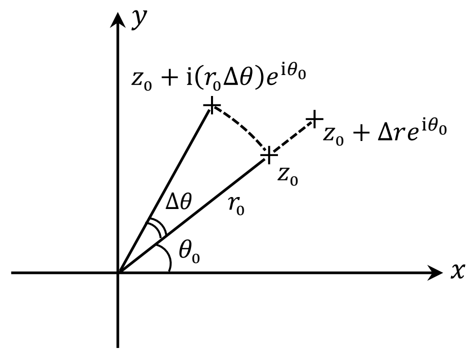
Problem 35) Moving in the radial direction by Δr, the derivative of $f(z)$ at z_0 is found to be



$$f'(z_0) = \frac{f(z_0+\Delta r e^{\mathrm{i}\theta_0}) - f(z_0)}{\Delta r e^{\mathrm{i}\theta_0}}$$

$$= \frac{u(r_0+\Delta r,\theta_0) + \mathrm{i}v(r_0+\Delta r,\theta_0) - u(r_0,\theta_0) - \mathrm{i}v(r_0,\theta_0)}{\Delta r e^{\mathrm{i}\theta_0}}$$

$$= e^{-\mathrm{i}\theta_0}\left[\frac{\partial u(r,\theta)}{\partial r} + \mathrm{i}\frac{\partial v(r,\theta)}{\partial r}\right]_{(r_0,\theta_0)}.$$

Next, we move in the azimuthal direction by $\Delta\theta$. The derivative of $f(z)$ at z_0 is now given by

$$f'(z_0) = \frac{f(z_0+\mathrm{i}r_0\Delta\theta e^{\mathrm{i}\theta_0}) - f(z_0)}{\mathrm{i}r_0\Delta\theta e^{\mathrm{i}\theta_0}}$$

$$= \frac{u(r_0,\theta_0+\Delta\theta) + \mathrm{i}v(r_0,\theta_0+\Delta\theta) - u(r_0,\theta_0) - \mathrm{i}v(r_0,\theta_0)}{\mathrm{i}r_0\Delta\theta e^{\mathrm{i}\theta_0}} = \frac{1}{\mathrm{i}r_0 e^{\mathrm{i}\theta_0}}\left[\frac{\partial u(r,\theta)}{\partial\theta} + \mathrm{i}\frac{\partial v(r,\theta)}{\partial\theta}\right]_{(r_0,\theta_0)}$$

$$= e^{-\mathrm{i}\theta_0}\left[\frac{1}{r_0}\frac{\partial v(r,\theta)}{\partial\theta} - \mathrm{i}\frac{1}{r_0}\frac{\partial u(r,\theta)}{\partial\theta}\right]_{(r_0,\theta_0)}.$$

The derivatives obtained by these two methods must be identical if the function is to be differentiable at $z = z_0$. Therefore,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta} \quad \leftarrow \text{Cauchy-Riemann conditions in polar coordinates.}$$

b) $f(z) = z^{1/2} = (re^{\mathrm{i}\theta})^{1/2} = \sqrt{r}e^{\mathrm{i}\theta/2}$; $\quad (r \ge 0, \quad 0 \le \theta < 2\pi)$.

$$\rightarrow \begin{cases} u(r,\theta) = \sqrt{r}\cos(\theta/2) & \rightarrow \quad \dfrac{\partial u}{\partial r} = \dfrac{\cos(\theta/2)}{2\sqrt{r}}, \quad \dfrac{\partial u}{\partial\theta} = -\dfrac{\sqrt{r}\sin(\theta/2)}{2}, \\ v(r,\theta) = \sqrt{r}\sin(\theta/2) & \rightarrow \quad \dfrac{\partial v}{\partial r} = \dfrac{\sin(\theta/2)}{2\sqrt{r}}, \quad \dfrac{\partial v}{\partial\theta} = \dfrac{\sqrt{r}\cos(\theta/2)}{2}. \end{cases}$$

The above derivatives are valid everywhere except at the origin, where $r = 0$, and on the positive real axis, where $\theta = 0$. The reason for the latter restriction is that $u(r,\theta)$ and $v(r,\theta)$ are discontinuous on the positive real axis and, therefore, cannot have a derivative there.

Checking the Cauchy-Riemann conditions:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta} \quad \rightarrow \quad \frac{\cos(\theta/2)}{2\sqrt{r}} = \frac{\sqrt{r}\cos(\theta/2)}{2r}.$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta} \quad \rightarrow \quad \frac{\sin(\theta/2)}{2\sqrt{r}} = \frac{\sqrt{r}\sin(\theta/2)}{2r}.$$

c) The branch-cut may be taken along any line that starts at the origin and goes to infinity. In the above discussion, this line was taken to be the positive real axis.