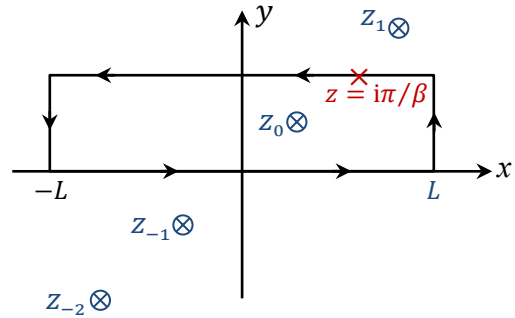


Problem 33) Taking advantage of the fact that the integrand is an even function of x , we extend the domain of integration from $(0, \infty)$ to $(-\infty, \infty)$. We also express $\cos(\alpha x)$ in terms of complex exponentials, and arrive at

$$\int_0^\infty \frac{\cos(\alpha x)}{\cosh(\beta x)} dx = \frac{1}{4} \int_{-\infty}^\infty \frac{\exp(i\alpha x)}{\cosh(\beta x)} dx + \frac{1}{4} \int_{-\infty}^\infty \frac{\exp(-i\alpha x)}{\cosh(\beta x)} dx. \quad (1)$$

To find the poles of the above integrands, we set $\cosh(\beta z)$ equal to zero, which yields $\exp(2\beta z) = -1 = \exp[i(2n + 1)\pi]$. Therefore, the poles are at $z_n = i(n + \frac{1}{2})\pi/\beta$, where $n = 0, \pm 1, \pm 2, \dots$. The integration contour for the integrals on the right-hand side of Eq.(1) is shown on the right. Next, we evaluate $\cosh(\beta z)$ on the upper leg of the rectangular contour by writing $z = x + i\eta$, where η is an arbitrary complex constant. We will have



$$\begin{aligned} \cosh(\beta z) &= \frac{1}{2}\{\exp[\beta(x + i\eta)] + \exp[-\beta(x + i\eta)]\} \\ &= \frac{1}{2} \exp(\beta x) [\cos(\beta\eta) + i \sin(\beta\eta)] + \frac{1}{2} \exp(-\beta x) [\cos(\beta\eta) - i \sin(\beta\eta)] \\ &= \cosh(\beta x) \cos(\beta\eta) + i \sinh(\beta x) \sin(\beta\eta). \end{aligned} \quad (2)$$

Therefore, on the upper leg of the rectangular contour, where $z = x + i(\pi/\beta)$, we have $\cosh(\beta z) = -\cosh(\beta x)$. On the vertical legs, it is not difficult to show that the integrands approach zero when $L \rightarrow \infty$. We thus have

$$\begin{aligned} \int_{-\infty}^\infty \frac{\exp(i\alpha x)}{\cosh(\beta x)} dx + \int_{-\infty}^\infty \frac{\exp[i\alpha(x+i\pi/\beta)]}{\cosh(\beta x)} dx &= 2\pi i \left[\frac{\exp(i\alpha z_0)}{\beta \sinh(\beta z_0)} \right] \leftarrow \text{Residue at } z_0 \\ &= 2\pi i \left[\frac{\exp(-\pi\alpha/2\beta)}{\beta \sinh(i\pi/2)} \right] = \left(\frac{2\pi}{\beta}\right) \exp(-\pi\alpha/2\beta). \end{aligned} \quad (3)$$

The above equation yields

$$\int_{-\infty}^\infty \frac{\exp(i\alpha x)}{\cosh(\beta x)} dx = \left(\frac{2\pi}{\beta}\right) \frac{\exp(-\pi\alpha/2\beta)}{1 + \exp(-\pi\alpha/\beta)} = \frac{\pi/\beta}{\cosh(\pi\alpha/2\beta)}. \quad (4)$$

Similarly, the second integral on the right-hand side of Eq.(1) is evaluated with the aid of the rectangular contour as follows:

$$\begin{aligned} \int_{-\infty}^\infty \frac{\exp(-i\alpha x)}{\cosh(\beta x)} dx + \int_{-\infty}^\infty \frac{\exp[-i\alpha(x+i\pi/\beta)]}{\cosh(\beta x)} dx &= 2\pi i \left[\frac{\exp(-i\alpha z_0)}{\beta \sinh(\beta z_0)} \right] \leftarrow \text{Residue at } z_0 \\ &= 2\pi i \left[\frac{\exp(\pi\alpha/2\beta)}{\beta \sinh(i\pi/2)} \right] = \left(\frac{2\pi}{\beta}\right) \exp(\pi\alpha/2\beta). \end{aligned} \quad (5)$$

Consequently,

$$\int_{-\infty}^\infty \frac{\exp(-i\alpha x)}{\cosh(\beta x)} dx = \left(\frac{2\pi}{\beta}\right) \frac{\exp(\pi\alpha/2\beta)}{1 + \exp(\pi\alpha/\beta)} = \frac{\pi/\beta}{\cosh(\pi\alpha/2\beta)}. \quad (6)$$

A final substitution into Eq.(1) yields

$$\int_0^\infty \frac{\cos(\alpha x)}{\cosh(\beta x)} dx = \frac{\pi/2\beta}{\cosh(\pi\alpha/2\beta)}. \quad (7)$$