Problem 32)

a) $\cosh(x + iy) = \frac{1}{2}\exp(x)\exp(iy) + \frac{1}{2}\exp(-x)\exp(-iy)$

$= \frac{1}{2}\exp(x)(\cos y + i\sin y) + \frac{1}{2}\exp(-x)(\cos y - i\sin y)$

$= \frac{1}{2}[\exp(x) + \exp(-x)]\cos(y) + \frac{1}{2}i[\exp(x) - \exp(-x)]\sin(y)$

$= \cosh(x)\cos(y) + i\sinh(x)\sin(y).$ (1)

b) $\cosh(z) = 0 \;\rightarrow\; \exp(z) + \exp(-z) = 0 \;\rightarrow\; \exp(2z) = -1 = \exp[\pm i(2n + 1)\pi].$

Consequently, the zeros of $\cosh(z)$ are at $z_n = \pm i(n + \frac{1}{2})\pi$, where $n = 0, 1, 2, 3, \cdots$.

Alternatively, one could use Eq.(1) and set $\cosh(x)\cos(y) = 0$, which yields $y_n = \pm(n + \frac{1}{2})\pi$, and $\sinh(x)\sin(y) = 0$, which yields $x = 0$. Therefore, $z_n = \pm i(n + \frac{1}{2})\pi$.

c) On the horizontal line $z = x + i\pi$ within the complex z-plane, we have

$$\cosh(z) = \cosh(x)\cos(\pi) + i\sinh(x)\sin(\pi) = -\cosh(x).$$ (2)

d) The integration contour in the complex z-plane is shown below. Considering that, on the short vertical legs of the contour at $x = \pm L$, the integrand $1/\cosh(z)$ goes to zero when $L \to \infty$, we will have

$$\int_{-\infty}^{\infty}\frac{dx}{\cosh x} - \int_{-\infty}^{\infty}\frac{dx}{-\cosh x} = 2\pi i \times [\text{Residue of } 1/\cosh(z) \text{ at } z_0 = i\pi/2].$$ (3)
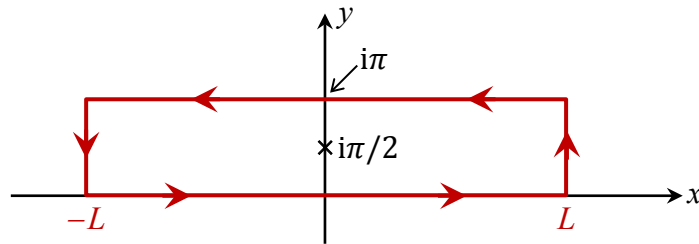
A Taylor series expansion of $\cosh(z)$ around the pole of the integrand at $z_0 = i\pi/2$ yields

$$\cosh(z) = \cosh(z_0) + \sinh(z_0)(z - z_0) + \frac{\cosh(z_0)}{2!}(z - z_0)^2 + \cdots$$

$$= 0 + \frac{1}{2}[\exp(i\pi/2) - \exp(-i\pi/2)](z - z_0) + 0 + \cdots$$

$$= i(z - z_0) + \text{higher order terms}.$$

The residue of $1/\cosh(z)$ at $z_0 = i\pi/2$ is thus seen to be $1/i$. Therefore, Eq.(3) may be written

$$2\int_{-\infty}^{\infty}\frac{dx}{\cosh x} = 2\pi i/i = 2\pi \qquad \rightarrow \qquad \int_{-\infty}^{\infty}\frac{dx}{\cosh x} = \pi.$$ (4)

The second integral, $\int_{-\infty}^{\infty}[x/\cosh(x)]dx$, obviously vanishes because its integrand is an odd function of x. Nevertheless, we proceed to use the Cauchy theorem on the same contour as before to obtain

$$\int_{-\infty}^{\infty}\frac{x}{\cosh x}dx - \int_{-\infty}^{\infty}\frac{x + i\pi}{-\cosh x}dx = 2\pi i \times [\text{Residue of } z/\cosh(z) \text{ at } z_0 = i\pi/2]$$

$$\rightarrow \quad 2\int_{-\infty}^{\infty}\frac{x}{\cosh x}dx + i\pi\int_{-\infty}^{\infty}\frac{1}{\cosh x}dx = 2\pi i \times \pi/2 \quad \rightarrow \quad \int_{-\infty}^{\infty}\frac{x}{\cosh x}dx = 0. \tag{5}$$

As for the third integral, $\int_{-\infty}^{\infty}[x^2/\cosh(x)]dx$, we use a similar procedure, namely,

$$\int_{-\infty}^{\infty}\frac{x^2}{\cosh x}dx - \int_{-\infty}^{\infty}\frac{(x + i\pi)^2}{-\cosh x}dx = 2\pi i \times [\text{Residue of } z^2/\cosh(z) \text{ at } z_0 = i\pi/2]$$

$$\rightarrow 2\int_{-\infty}^{\infty}\frac{x^2\,dx}{\cosh x} + i2\pi\int_{-\infty}^{\infty}\frac{x\,dx}{\cosh x}^{\;0} - \pi^2\int_{-\infty}^{\infty}\frac{dx}{\cosh x} = 2\pi \times (i\pi/2)^2 \quad \rightarrow \quad \int_{-\infty}^{\infty}\frac{x^2\,dx}{\cosh x} = \pi^3/4. \tag{6}$$