Problem 31) Considering the closed contour of integration shown in the figure, the integrand $f(z)\ln z$ along the straight line immediately above the x-axis equals $f(x)\ln x$. However, on the straight line immediately below the x-axis, $f(z)\ln z = f(x)(\ln x + \mathrm{i}2\pi)$. The contribution of the two straight lines to the loop integral, therefore, equals $-\mathrm{i}2\pi\int_{x=0}^{\infty} f(x)\mathrm{d}x$.

Next, we examine the small circle of radius ε surrounding the origin, around which $f(z)\ln z = f(\varepsilon e^{\mathrm{i}\theta})(\ln\varepsilon + \mathrm{i}\theta)$. The contribution of this circle to the loop integral is given by

$$\oint_{\text{small circle}} f(z)\ln z\,\mathrm{d}z = \int_{\theta=0}^{2\pi} f(\varepsilon e^{\mathrm{i}\theta})(\ln\varepsilon + \mathrm{i}\theta)\underbrace{\mathrm{i}\varepsilon e^{\mathrm{i}\theta}\mathrm{d}\theta}_{\mathrm{d}z}.$$

Now, in the limit when $\varepsilon \to 0$, $\varepsilon\ln\varepsilon \to 0$ and, given that $f(z)$ is analytic at $z = 0$, in the immediate neighborhood of the origin, $f(\varepsilon e^{\mathrm{i}\theta})$ approaches $f(0)$. Consequently, the entire integral over the small circle vanishes when $\varepsilon \to 0$. Also, by assumption, the integral over the large circle approaches zero when $R \to \infty$. All in all, the integral around the closed loop reduces to $-\mathrm{i}2\pi\int_{x=0}^{\infty} f(x)\mathrm{d}x$, which, in accordance with the Cauchy-Goursat theorem must equal $2\pi\mathrm{i}$ times the sum of all residues of $f(z)\ln z$. Therefore,

$$\int_{x=0}^{\infty} f(x)\mathrm{d}x = -\sum\nolimits_{\text{all singularities}} \text{Residues of } f(z)\ln z.$$

a) The function $f(z)\ln z$ in this case is given by

$$\frac{\ln z}{z^3 + a^3} = \frac{\ln z}{(z+a)[z - a\exp(\mathrm{i}\pi/3)][z - a\exp(\mathrm{i}5\pi/3)]}.$$

The poles of the function are specified as $z_1 = -a = a\exp(\mathrm{i}\pi)$, $z_2 = a\exp(\mathrm{i}\pi/3)$, and $z_3 = a\exp(\mathrm{i}5\pi/3)$, so that their phase angles fall within the interval $[0, 2\pi)$, consistent with our choice of the branch-cut for $\ln z$. Thus, the sum of the residues of $f(z)\ln z$ within the specified contour is determined as follows:

$$\begin{aligned}\sum \text{Residues} &= \frac{(\ln a) + \mathrm{i}\pi}{[-a - a\exp(\mathrm{i}\pi/3)][-a - a\exp(5\mathrm{i}\pi/3)]} + \frac{(\ln a) + \mathrm{i}(\pi/3)}{[a\exp(\mathrm{i}\pi/3) + a][a\exp(\mathrm{i}\pi/3) - a\exp(\mathrm{i}5\pi/3)]} \\ &\quad + \frac{(\ln a) + \mathrm{i}(5\pi/3)}{[a\exp(\mathrm{i}5\pi/3) + a][a\exp(\mathrm{i}5\pi/3) - a\exp(\mathrm{i}\pi/3)]} \\ &= \frac{(\ln a) + \mathrm{i}\pi}{a^2[1 + \exp(\mathrm{i}\pi/3)][1 + \exp(\mathrm{i}5\pi/3)]} + \frac{(\ln a) + \mathrm{i}(\pi/3)}{a^2[1 + \exp(\mathrm{i}\pi/3)][\exp(\mathrm{i}\pi/3) - \exp(\mathrm{i}5\pi/3)]} \\ &\quad + \frac{(\ln a) + \mathrm{i}(5\pi/3)}{a^2[1 + \exp(\mathrm{i}5\pi/3)][\exp(\mathrm{i}5\pi/3) - \exp(\mathrm{i}\pi/3)]} \\ &= \frac{[(\ln a)+\mathrm{i}\pi][\exp(\mathrm{i}\pi/3) - \exp(\mathrm{i}5\pi/3)] + [(\ln a)+\mathrm{i}(\pi/3)][1 + \exp(\mathrm{i}5\pi/3)] - [(\ln a)+\mathrm{i}(5\pi/3)][1 + \exp(\mathrm{i}\pi/3)]}{a^2[1 + \exp(\mathrm{i}\pi/3)][1 + \exp(\mathrm{i}5\pi/3)][\exp(\mathrm{i}\pi/3) - \exp(\mathrm{i}5\pi/3)]} \\ &= \frac{\mathrm{i}\pi[\exp(\mathrm{i}\pi/3) - \exp(\mathrm{i}5\pi/3)] + \mathrm{i}(\pi/3)[\exp(\mathrm{i}5\pi/3) - 5\exp(\mathrm{i}\pi/3) - 4]}{a^2[2 + \exp(\mathrm{i}\pi/3) + \exp(\mathrm{i}5\pi/3)][\exp(\mathrm{i}\pi/3) - \exp(\mathrm{i}5\pi/3)]} \\ &= \frac{\mathrm{i}\pi(\mathrm{i}\sqrt{3}) + \mathrm{i}(\pi/3)\left(\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2} - \frac{5}{2} - \mathrm{i}\frac{5\sqrt{3}}{2} - 4\right)}{\mathrm{i}3\sqrt{3}a^2} = \frac{-\sqrt{3}\pi - \mathrm{i}\pi(2 + \mathrm{i}\sqrt{3})}{\mathrm{i}3\sqrt{3}a^2} = -\frac{2\pi\sqrt{3}}{9a^2}.\end{aligned}$$

b) The function $f(z)\ln z$ in this case is given by

$$f(z)\ln z = \frac{\ln z}{(z+a)(z^2+b^2)} = \frac{\ln z}{(z+a)(z-\mathrm{i}b)(z+\mathrm{i}b)}$$

The poles of the function are specified as $z_1 = -a = a\exp(\mathrm{i}\pi)$, $z_2 = \mathrm{i}b = b\exp(\mathrm{i}\pi/2)$, and $z_3 = -\mathrm{i}b = b\exp(\mathrm{i}3\pi/2)$, so that their phase angles fall within the interval $[0, 2\pi)$, consistent with our choice of the branch-cut for $\ln z$. Thus, the sum of the residues of $f(z)\ln z$ within the specified contour is determined as follows:

$$\begin{aligned}
\sum \text{Residues} &= \frac{(\ln a)+\mathrm{i}\pi}{a^2+b^2} + \frac{(\ln b)+\mathrm{i}(\pi/2)}{(\mathrm{i}b+a)(2\mathrm{i}b)} + \frac{(\ln b)+\mathrm{i}(3\pi/2)}{(-\mathrm{i}b+a)(-2\mathrm{i}b)} \\
&= \frac{(\ln a)+\mathrm{i}\pi}{a^2+b^2} + \frac{[(\ln b)+\mathrm{i}(\pi/2)](\mathrm{i}a+b) - [(\ln b)+\mathrm{i}(3\pi/2)](\mathrm{i}a-b)}{2b(\mathrm{i}a-b)(\mathrm{i}a+b)} \\
&= \frac{(\ln a)+\mathrm{i}\pi}{a^2+b^2} - \frac{(2b\ln b)+\pi(a+2\mathrm{i}b)}{2b(a^2+b^2)} = \frac{(\ln a)+\mathrm{i}\pi}{a^2+b^2} - \frac{(\ln b)+\mathrm{i}\pi}{a^2+b^2} - \frac{\pi a}{2b(a^2+b^2)} \\
&= -\frac{\pi a + 2b\ln(b/a)}{2b(a^2+b^2)}.
\end{aligned}$$
