Problem 30) The integral under consideration may be written as follows:

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{(x+b)^2+c^2}\,\mathrm{d}x = ½\int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}ax)}{(x+b)^2+c^2}\,\mathrm{d}x + ½\int_{-\infty}^{\infty} \frac{\exp(-\mathrm{i}ax)}{(x+b)^2+c^2}\,\mathrm{d}x. \quad (1)$$

Writing the integrands in the complex z-plane as $\exp(\pm\mathrm{i}az)/[(z+b)^2+c^2]$, we identify the poles of these integrands as $z_{1,2} = -b \pm \mathrm{i}c$. On the right-hand side of Eq.(1), the first integral must be closed on a semi-circle in the upper-half of the z-plane, whereas the second integral must be closed on a semi-circle in the lower-half of the z-plane. For each integral, a single first-order pole resides inside the semi-circular contour. Therefore,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{(x+b)^2+c^2}\,\mathrm{d}x = 2\pi\mathrm{i}\times\frac{1}{2}\left[\frac{\exp(\mathrm{i}az_1)}{z_1-z_2} - \frac{\exp(-\mathrm{i}az_2)}{z_2-z_1}\right] = \mathrm{i}\pi\left\{\frac{\exp[\mathrm{i}a(-b+\mathrm{i}c)]}{2\mathrm{i}c} + \frac{\exp[-\mathrm{i}a(-b-\mathrm{i}c)]}{2\mathrm{i}c}\right\}$$

$$= (\pi/c)\exp(-ac)\cos(ab). \quad (2)$$

Differentiating both sides of the above identity with respect to a now yields

$$\frac{\mathrm{d}}{\mathrm{d}a}\int_{-\infty}^{\infty} \frac{\cos(ax)}{(x+b)^2+c^2}\,\mathrm{d}x = -\int_{-\infty}^{\infty} \frac{x\sin(ax)}{(x+b)^2+c^2}\,\mathrm{d}x$$

$$= (\pi/c)[-c\exp(-ac)\cos(ab) - b\exp(-ac)\sin(ab)]. \quad (3)$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{x\sin(ax)}{(x+b)^2+c^2}\,\mathrm{d}x = (\pi/c)\exp(-ac)\,[c\cos(ab) + b\sin(ab)]. \quad (4)$$

Next, we differentiate both sides of Eq.(2) with respect to c to arrive at

$$\frac{\mathrm{d}}{\mathrm{d}c}\int_{-\infty}^{\infty} \frac{\cos(ax)}{(x+b)^2+c^2}\,\mathrm{d}x = -2c\int_{-\infty}^{\infty} \frac{\cos(ax)}{[(x+b)^2+c^2]^2}\,\mathrm{d}x$$

$$= [-(\pi/c^2)\exp(-ac) - a(\pi/c)\exp(-ac)]\cos(ab)$$

$$= -(\pi/c^2)(1+ac)\exp(-ac)\cos(ab). \quad (5)$$

Consequently,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{[(x+b)^2+c^2]^2}\,\mathrm{d}x = \left(\frac{\pi}{2c^3}\right)(1+ac)\exp(-ac)\cos(ab). \quad (6)$$