Problem 29) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2-2x\cos\omega+1)}$; ω real and $\sin\omega \neq 0$.

The contour of integration is an infinitely large semi-circle in the upper half-plane. The poles are $z_1 = +i$, $z_2 = -i$, and also $z^2 - 2z\cos\omega + 1 = 0 \Rightarrow z = \cos\omega \pm \sqrt{\cos^2\omega - 1} = \cos\omega \pm i\sin\omega = e^{\pm i\omega}$, so that $z_3 = e^{+i\omega}$ and $z_4 = e^{-i\omega}$. Of these z_1 and z_3 are inside the contour (assuming $0<\omega<\pi$), while z_2 and z_4 are outside the contour.

$$\text{Residue at } z_1 = \frac{z_1^2}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} = \frac{(i)^2}{(2i)(i-e^{i\omega})(i-e^{-i\omega})}$$

$$= \frac{i}{2(\cancel{-1}+\cancel{1}-2i\cos\omega)} = -\frac{1}{4\cos\omega}. \checkmark$$

$$\text{Residue at } z_3 = \frac{z_3^2}{(z_3-z_1)(z_3-z_2)(z_3-z_4)} = \frac{e^{2i\omega}}{(e^{i\omega}-i)(e^{i\omega}+i)(e^{i\omega}-e^{-i\omega})}$$

$$= \frac{e^{2i\omega}}{(e^{2i\omega}+1)(2i\sin\omega)} = \frac{e^{i\omega}}{4i\sin\omega\cos\omega} = \frac{1}{4i\sin\omega} + \frac{1}{4\cos\omega}. \checkmark$$

$$\text{Therefore, } \int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2-2x\cos\omega+1)} = 2\pi i\left(\cancel{-\frac{1}{4\cos\omega}} + \frac{1}{4i\sin\omega} + \cancel{\frac{1}{4\cos\omega}}\right)$$

$$= \frac{\pi}{2\sin\omega}; \quad 0<\omega<\pi$$

Similarly, it can be shown that, if $\pi<\omega<2\pi$, then z_1 and z_4 will be the only poles inside the contour. In that case the integral will be $-\dfrac{\pi}{2\sin\omega}$. Thus, in general, the value of the integral will be

$$\frac{\pi}{2|\sin\omega|}.$$