Problem 28) $\int_0^\infty \frac{\cos ax}{x^4+4b^4}dx = \frac{1}{4}\int_{-\infty}^{\infty}\frac{e^{iax}}{x^4+4b^4}dx + \frac{1}{4}\int_{-\infty}^{\infty}\frac{e^{-iax}}{x^4+4b^4}dx$

For the first integral we use an infinitely large semi-circular contour in the upper half-plane, where Jordan's lemma can be applied. For the second integral a similar contour in the lower half-plane will be used. The poles are obtained as follows:

$$z^4+4b^4=0 \Rightarrow z^2=\pm i2b^2 = 2b^2e^{\pm i\pi/2} \Rightarrow z=\pm\sqrt{2}\,b\,e^{\pm i\pi/4}.$$

The poles $z_1=\sqrt{2}\,b\,e^{i\pi/4}$ and $z_2=-\sqrt{2}\,b\,e^{-i\pi/4}$ are in the upper-half plane, while $z_3=-\sqrt{2}\,b\,e^{i\pi/4}$ and $z_4=\sqrt{2}\,b\,e^{-i\pi/4}$ are in the lower half.

$$\text{Residue at } z_1 = \frac{e^{iaz_1}}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} = \frac{e^{ia\sqrt{2}b(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})}}{(2b)(2\sqrt{2}be^{i\pi/4})(2ib)}$$

$$= \frac{e^{-ab}\,e^{iab}}{i\,8\sqrt{2}\,e^{i\pi/4}\,b^3}$$

$$\text{Residue at } z_2 = \frac{e^{iaz_2}}{(z_2-z_1)(z_2-z_3)(z_2-z_4)} = \frac{e^{-ia\sqrt{2}b(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4})}}{(-2b)(2ib)(-2\sqrt{2}be^{-i\pi/4})}$$

$$= \frac{e^{-ab}\,e^{-iab}}{i\,8\sqrt{2}\,e^{-i\pi/4}\,b^3}$$

$$\text{Residue at } z_3 = -\frac{e^{-iaz_3}}{(z_3-z_1)(z_3-z_2)(z_3-z_4)} = \frac{-e^{+ia\sqrt{2}b(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})}}{(-2\sqrt{2}be^{i\pi/4})(-2ib)(-2b)}$$

$$= \frac{e^{-ab}\,e^{iab}}{i\,8\sqrt{2}\,e^{i\pi/4}\,b^3}$$

$$\text{Residue at } z_4 = -\frac{e^{-iaz_4}}{(z_4-z_1)(z_4-z_2)(z_4-z_3)} = \frac{-e^{-ia\sqrt{2}b(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4})}}{(-2ib)(2\sqrt{2}be^{-i\pi/4})(2b)}$$

$$= \frac{e^{-ab}\,e^{-iab}}{i\,8\sqrt{2}\,e^{-i\pi/4}\,b^3}$$

We now use Cauchy's theorem to write:

$$\int_0^\infty \frac{\cos(ax)}{x^4+4b^4}\,dx = \frac{1}{4}(2\pi i)\frac{e^{-ab}}{i\,8\sqrt{2}\,b^3}\left(e^{-i\pi/4}e^{iab}+e^{i\pi/4}e^{-iab}+e^{-i\pi/4}e^{iab}+e^{i\pi/4}e^{-iab}\right)$$

$$= \frac{\pi e^{-ab}}{16\sqrt{2}\,b^3}\left[2e^{iab}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)+2e^{-iab}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]$$

$$= \frac{\pi e^{-ab}}{16\,b^3}\left(2\cos ab+2\sin ab\right) = \frac{\pi}{8b^3}e^{-ab}(\cos ab+\sin ab).$$ ✓

Next, we differentiate both sides of the above identity with respect to b to obtain:

$$\frac{d}{db}\int_0^\infty \frac{\cos ax}{x^4+4b^4}\,dx = \int_0^\infty \frac{-16b^3\cos ax}{(x^4+4b^4)^2}\,dx = \left(\frac{-3\pi}{8b^4}e^{-ab}-\frac{\pi a}{8b^3}e^{-ab}\right)(\cos ab$$

$$+\sin ab)+\frac{\pi}{8b^3}e^{-ab}(-a\sin ab+a\cos ab) = -\frac{3\pi}{8b^4}e^{-ab}(\cos ab+\sin ab)$$

$$-\frac{\pi a}{8b^3}e^{-ab}(2\sin ab) \Longrightarrow$$

$$\int_0^\infty \frac{\cos ax}{(x^4+4b^4)^2}\,dx = \frac{3\pi}{128b^7}e^{-ab}(\cos ab+\sin ab)+\frac{\pi a}{64b^6}e^{-ab}\sin ab.$$