Problem 23) a) On the large circle the integral goes to zero as $R \to \infty$.

The reason being that $\left| \oint_{\text{Circle } R} \frac{z^{-a}}{1+z} dz \right| = \left| \int_{\theta=0}^{2\pi} \frac{iRe^{i\theta} d\theta}{R^a e^{ia\theta}(1+Re^{i\theta})} \right|$

$$\leq \int_0^{2\pi} \frac{R\,d\theta}{R^a |1+Re^{i\theta}|} \leq \int_0^{2\pi} \frac{R\,d\theta}{R^a(R-1)} = \frac{2\pi R}{R^a(R-1)} \xrightarrow[R\to\infty]{} 0 \qquad (a>0).$$

On the small circle surrounding $z_0=-1$, let $z=-1+\epsilon e^{i\theta}$. Then

$$\oint_{\substack{\text{Small Circle}\\ \text{at } z_0=-1}} \frac{z^{-a}\,dz}{z+1} = -\int_{\theta=0}^{2\pi} \frac{(-1+\epsilon e^{i\theta})^{-a}}{\cancel{\epsilon e^{i\theta}}} i\cancel{\epsilon e^{i\theta}}\, d\theta \xrightarrow[\epsilon\to 0]{} -2\pi i(-1)^{-a} = -2\pi i e^{-i\pi a}.$$

The minus sign in front of the integral accounts for the fact that the small circle is traversed clockwise.

On the small circle surrounding $z_0=0$, let $z=\epsilon e^{i\theta}$. We'll have

$$\oint_{\substack{\text{Small Circle}\\ \text{at } z_0=0}} \frac{\epsilon^{-a} e^{-ia\theta}}{(1+\epsilon e^{i\theta})} i\epsilon e^{i\theta} d\theta = -\int_{\theta=0}^{2\pi} \frac{i\epsilon^{1-a}}{1+\epsilon e^{i\theta}} e^{i(1-a)\theta} d\theta \xrightarrow[\epsilon\to 0]{} 0.$$

Clockwise rotation

(because $a<1$, yielding $\epsilon^{1-a} \to 0$)

On the straight-line going from $x=0$ to ∞ above the real axis,

$$z^{-a} = e^{-a\ln z} = e^{-a(\ln x + i0)} = e^{-a\ln x} = x^{-a}.$$

However, on the straight line from $x=0$ to ∞ below the real axis,

$$z^{-a} = e^{-a\ln z} = e^{-a(\ln x + i2\pi)} = e^{-a\ln x} e^{-i2\pi a} = x^{-a} e^{-i2\pi a}.$$

The total integral over the entire contour, which must be equal to zero because there are no poles inside, is thus given by:

$$\int_0^\infty \frac{x^{-a}}{x+1} dx - e^{-i2\pi a} \int_0^\infty \frac{x^{-a}}{x+1} dx - 2\pi i e^{-i\pi a} = 0 \Rightarrow$$

$$\int_0^\infty \frac{x^{-a}}{x+1} dx = \frac{2\pi i e^{-i\pi a}}{1-e^{-i2\pi a}} = \frac{2\pi i}{e^{i\pi a} - e^{-i\pi a}} = \frac{\pi}{\sin(\pi a)} \checkmark$$

b) This problem is similar to that in part (a). Again the integral over the large circle goes to zero when $R \to \infty$, this time because

$$\frac{2\pi R^{1+a}}{(R-1)^2} \underset{R\to\infty}{\longrightarrow} 0$$ as long as $a<1$. The integral over the small circle surrounding $z_0=0$ is zero once again, because $\epsilon^{1+a} \underset{\epsilon\to 0}{\longrightarrow} 0$ provided that $a>-1$. The integral over the small circle at $z_0=-1$ is given by:

$$\oint_{\substack{\text{small circle}\\ \text{at } z_0=-1}} \frac{z^a}{(z+1)^2}\,dz = -\int_{\theta=0}^{2\pi} \frac{z^a}{(\epsilon e^{i\theta})^2}\, i\epsilon e^{i\theta}\,d\theta \leftarrow z=-1+\epsilon e^{i\theta}$$

Clockwise rotation

$$= -i\int_0^{2\pi} \frac{z^a}{\epsilon e^{i\theta}}\,d\theta = -i\int_0^{2\pi} \frac{(-1+\epsilon e^{i\theta})^a}{\epsilon e^{i\theta}}\,d\theta = -i\int_0^{2\pi} \frac{(-1)^a + a(-1)^{a-1}\epsilon e^{i\theta}+\cdots}{\epsilon e^{i\theta}}\,d\theta$$

$$= -i\left\{ \frac{(-1)^a}{\epsilon}\cancelto{0}{\int_0^{2\pi} e^{-i\theta}\,d\theta} + a(-1)^{a-1}\int_0^{2\pi} d\theta + O(\epsilon)\right\} \underset{\epsilon\to 0}{\longrightarrow} -i2\pi a e^{i(a-1)\pi}$$

The integral over the straight line from $x=0$ to ∞ above the real-axis is just $\int_0^\infty \frac{x^a}{(x+1)^2}\,dx$. The integral over the straight line from $x=0$ to ∞ below the real-axis is $e^{i2\pi a}\int_0^\infty \frac{x^a}{(x+1)^2}\,dx$. Combining these results and noting that the integral over the entire contour must be zero (because there are no poles inside), we'll have:

$$\int_0^\infty \frac{x^a\,dx}{(x+1)^2} - e^{i2\pi a}\int_0^\infty \frac{x^a}{(x+1)^2}\,dx - i2\pi a e^{i(a-1)\pi} = 0 \Rightarrow$$

$$\int_0^\infty \frac{x^a\,dx}{(x+1)^2} = \frac{i2\pi a e^{-i\pi} e^{ia\pi}}{1-e^{i2\pi a}} = \frac{-i2\pi a}{e^{-i\pi a} - e^{+i\pi a}} = \frac{\pi a}{\sin(\pi a)}. \checkmark$$