Problem 22) The function $\ln(t)$ has an interesting symmetry with respect to the point $t=1$. To see this, write the integral as the sum of two integrals, one over the range $t\in[0,1]$, the other over the range $t\geq 1$. We find:

$$\int_0^\infty \frac{(\ln t)^2}{1+t^2}dt = \int_0^1 \frac{(\ln t)^2}{1+t^2}dt + \int_1^\infty \frac{(\ln t)^2}{1+t^2}dt$$

Change of variable $t'=\frac{1}{t} \Rightarrow dt' = -\frac{dt}{t^2}$ $\Rightarrow dt = -\frac{dt'}{t'^2}$. Also $(\ln t)^2 = \ln^2(\frac{1}{t'}) = (-\ln t')^2 = \ln^2 t'$.

$$= \int_0^1 \frac{\ln^2 t}{1+t^2}dt - \int_1^0 \frac{\ln^2 t'}{(1+\frac{1}{t'^2})}\frac{dt'}{t'^2}$$

$$= \int_0^1 \frac{\ln^2 t}{1+t^2}dt + \int_0^1 \frac{\ln^2 t'}{1+t'^2}dt' = 2\int_0^1 \frac{\ln^2 t}{1+t^2}dt.$$

Next, we expand $\frac{1}{1+t^2}$ into the geometric series $1-t^2+t^4-t^6+\dots = \sum_{n=0}^\infty (-1)^n t^{2n}$, change the order of summation and integration, and use integration by parts (twice) to evaluate the individual integrals. We'll have:

$$\int_0^\infty \frac{\ln^2 t}{1+t^2}dt = 2\int_0^1 \ln^2(t)\sum_{n=0}^\infty (-1)^n t^{2n}dt = 2\sum_{n=0}^\infty (-1)^n \int_0^1 t^{2n}\ln^2 t\, dt$$

$$= 2\sum_{n=0}^\infty (-1)^n \left\{ \underbrace{\frac{t^{2n+1}}{2n+1}\ln^2 t\Big|_0^1}_{0} - \int_0^1 \frac{t^{2n+1}}{2n+1}\frac{2\ln t}{t}dt \right\}$$

$$= -4\sum_{n=0}^\infty \frac{(-1)^n}{2n+1}\int_0^1 t^{2n}\ln t\, dt = -4\sum_{n=0}^\infty \frac{(-1)^n}{2n+1}\left\{ \underbrace{\frac{t^{2n+1}}{2n+1}\ln t\Big|_0^1}_{0} - \int_0^1 \frac{t^{2n+1}}{2n+1}\frac{dt}{t}\right\}$$

$$= 4\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^2}\int_0^1 t^{2n}dt = 4\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^3} t^{2n+1}\Big|_0^1 = 4\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^3}.$$

Note that in the above derivation we have used the fact that $t\ln^2 t$ and $t\ln t$ approach zero as $t\to 0$.

Next, we change the variable from "t" to $x = \ln t$, using $dx = \frac{dt}{t} = \frac{dt}{e^x}$ to write:

$$\int_0^\infty \frac{\ln^2 t}{1+t^2}\,dt = \int_{-\infty}^{\infty} \frac{x^2}{1+e^{2x}}\, e^x dx = \int_{-\infty}^{\infty} \frac{x^2}{e^{-x}+e^{x}}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2 dx}{\cosh x}$$

To do the contour integral we follow several steps:

1) On the vertical legs of the contour, $\frac{z^2}{\cosh z} \xrightarrow[R\to\infty]{} 0$ because $\cosh z$ behaves like e^x when $x \to \infty$, and like e^{-x} when $x \to -\infty$. In both cases the denominator ($\cosh z$) grows exponentially with R, while the numerator (z^2) goes up only with the 2nd power of z. Therefore, the vertical legs of the contour do <u>not</u> contribute to the integral.

2) On the upper leg of the contour $z^2 = (x+i\pi)^2 = x^2 - \pi^2 + 2i\pi x$.
Also $\cosh(z) = \frac{1}{2}\left(e^{x+i\pi} + e^{-x-i\pi}\right) = \frac{1}{2}\left(e^x e^{i\pi} + e^{-x}e^{-i\pi}\right) = \frac{1}{2}\left(-e^x - e^{-x}\right)$
$= -\cosh(x)$. Finally $dz = d(x+i\pi) = dx$. Therefore,

$$\int_{\text{upper leg}} \frac{z^2}{2\cosh(z)}\,dz = \int_{+\infty}^{-\infty} \frac{(x^2-\pi^2+2i\pi x)}{-2\cosh(x)}\,dx = \int_{-\infty}^{\infty} \frac{x^2-\pi^2+2i\pi x}{2\cosh x}\,dx.$$

Now $2i\pi x$ is an odd function of x, while $\cosh x$ is even, so the integral of $\frac{2i\pi x}{2\cosh x} \to 0$. The term $-\frac{\pi^2}{2\cosh x}$ can be readily integrated as follows:

$$\int_{-\infty}^{\infty} \frac{-\pi^2}{2\cosh x}\,dx = -\pi^2 \int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}} =$$

Change of Variable $y = e^{+x}$

$$= -\pi^2\int_0^\infty \frac{dy}{\left(y+\frac{1}{y}\right)y} = -\pi^2\int_0^\infty \frac{dy}{1+y^2} = -\pi^2\int_0^{\pi/2} \frac{\cancel{(1+\tan^2\theta)}\,d\theta}{\cancel{1+\tan^2\theta}} = -\frac{\pi^3}{2}$$

Change of Variable $y = \tan\theta$

The integral around the rectangular loop is thus seen to be equal to $\underbrace{\int_{-\infty}^{\infty} \frac{x^2 dx}{2\cosh x}}_{\text{lower leg}} + \underbrace{\int_{-\infty}^{\infty} \frac{x^2 dx}{2\cosh x} - \frac{\pi^3}{2}}_{\text{upper leg}} = 2\int_{-\infty}^{\infty} \frac{x^2 dx}{2\cosh x} - \frac{\pi^3}{2}$.

The Contour integral must be equal to $2\pi i$ times the sum of the residues at the poles of the integrand located inside the Contour. The only pole of the integrand, $\frac{z^2}{2\cosh z}$, inside the loop is $z_0 = \frac{i}{2}\pi$ (See problem 25d). To find the residue at this pole, we expand $\cosh z$ in a Taylor series around $z = z_0 = i\pi/2$, as follows:

$$\cosh z = \cosh(z_0) + \sinh(z_0)(z - z_0) + \cosh(z_0)\frac{(z-z_0)^2}{2!} + \cdots$$

$$= \cosh\left(\frac{i\pi}{2}\right) + \sinh\left(\frac{i\pi}{2}\right)\left(z - \frac{i\pi}{2}\right) + \cdots$$

$$= 0 \quad + \left(\frac{e^{i\pi/2} - e^{-i\pi/2}}{2}\right)\left(z - \frac{i\pi}{2}\right) + \cdots$$

$$\simeq i\left(z - \frac{i\pi}{2}\right) \quad \leftarrow \text{when } z \to i\pi/2.$$

In the vicinity of the pole at $z_0 = i\pi/2$, the integrand is approximated as $\frac{z^2}{2\cosh(z)} \simeq \frac{z_0^2}{2i(z - z_0)} = \frac{(i\pi/2)^2}{2i(z - i\pi/2)} = \frac{i\pi^2/8}{z - i\pi/2}$. The Residue is therefore $i\pi^2/8$, which, after multiplication with $2\pi i$, yields the Contour integral as $-\pi^3/4$. Setting this equal to the value of the integral that was previously evaluated, we find:

$$2\int_{-\infty}^{\infty} \frac{x^2}{2\cosh x}dx - \frac{\pi^3}{2} = -\frac{\pi^3}{4} \implies \int_0^{\infty} \frac{\ln^2 t}{1+t^2}dt = \int_{-\infty}^{\infty} \frac{x^2}{2\cosh x}dx = \frac{\pi^3}{8}$$