Problem 20) On the circular arc the integral goes to zero as $R \to \infty$

because $\left| \int_{arc} \frac{dz}{1+z^n} \right| = \left| \int_{\theta=0}^{2\pi/n} \frac{iRe^{i\theta}d\theta}{1+R^n e^{in\theta}} \right| \le \int_0^{2\pi/n} \frac{R\,d\theta}{|1+R^n e^{in\theta}|} \le \frac{R}{R^n-1} \int_0^{2\pi/n} d\theta \xrightarrow[R\to\infty]{} 0$.

On the straight-line oriented at $\theta = 2\pi/n$, we have $z = re^{i2\pi/n}$,

$1+z^n = 1+r^n e^{i2\pi} = 1+r^n$, and $dz = e^{i2\pi/n}dr$. Therefore,

$$\int_{\substack{\text{Straight}\\ \text{line at } 2\pi/n}} \frac{dz}{1+z^n} = \int_0^\infty \frac{e^{i2\pi/n}dr}{1+r^n} = e^{i2\pi/n}\int_0^\infty \frac{dx}{1+x^n}.$$

Finally, the poles of the integrand are at $1+z^n=0 \Rightarrow z^n = e^{i(2m+1)\pi}$

$\Rightarrow z = e^{i(2m+1)\pi/n}$. The only one of these poles inside the closed contour is $e^{i\pi/n}$. To find the residue at this pole, we use the result obtained in Problem 16, namely $\frac{d}{dz}(1+z^n)\Big|_{z=e^{i\pi/n}} = n(e^{i\pi/n})^{n-1}$

$= ne^{i\pi(1-\frac{1}{n})} = -ne^{-i\pi/n}$. The desired residue is, therefore, given by $\frac{1}{-ne^{-i\pi/n}} = -\frac{e^{i\pi/n}}{n}$. Combining these results, we'll have:

$$\int_0^\infty \frac{dx}{1+x^n} - e^{i2\pi/n}\int_0^\infty \frac{dx}{1+x^n} = 2\pi i\left(-\frac{e^{i\pi/n}}{n}\right)$$

$$\Rightarrow \int_0^\infty \frac{dx}{1+x^n} = -\frac{2\pi i}{n}\frac{e^{i\pi/n}}{1-e^{i2\pi/n}} = -\frac{2\pi i}{n}\frac{1}{e^{-i\pi/n}-e^{i\pi/n}} = \frac{\pi/n}{\sin(\pi/n)}.$$

These results are valid when n is an integer greater than or equal to 2.