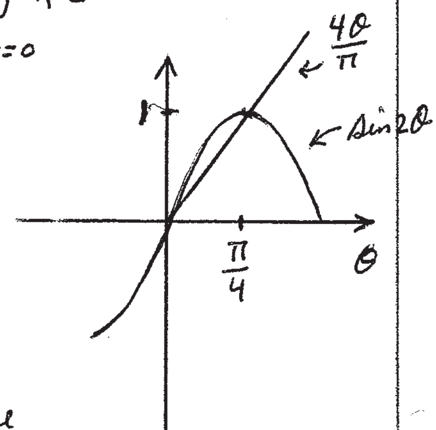
Problem 14) The integral to evaluate is $\int_0^\infty e^{ix^2}dx$, whose real and imaginary parts will then yield the desired integrals, $\int_0^\infty \cos(x^2)dx$ and $\int_0^\infty \sin(x^2)dx$. On the circular arc the integral of e^{iz^2} goes to zero as $R\to\infty$. This is because

$$\left|\int_{arc} e^{iz^2}dz\right| = \left|\int_{arc} e^{i(R^2\cos 2\theta + iR^2\sin 2\theta)}\, iRe^{i\theta}d\theta\right| \le \int_{\theta=0}^{\pi/4} Re^{-R^2\sin 2\theta}d\theta$$

$$< \int_{\theta=0}^{\pi/4} Re^{-(4R^2/\pi)\theta}d\theta = \frac{\pi}{4R}\left(1-e^{-R^2}\right) \xrightarrow[R\to\infty]{} 0.$$



Consequently, since e^{iz^2} has no poles, the integral on the x-axis will be equal to the integral on the 45° line, that is, the line on which $z = re^{i\pi/4}$, $e^{iz^2} = e^{ir^2e^{i\pi/2}} = e^{-r^2}$, and $dz = e^{i\pi/4}dr$. We'll have:

$$\int_0^\infty e^{ix^2}dx = \int_{r=0}^\infty e^{-r^2}e^{i\pi/4}dr = e^{i\pi/4}\int_0^\infty e^{-r^2}dr$$

We know (from Problem 4) how to evaluate $\int_{-\infty}^\infty e^{-x^2}dx$, namely,

$$\left(\int_{-\infty}^\infty e^{-x^2}dx\right)^2 = \iint_{-\infty}^\infty e^{-(x^2+y^2)}dx\,dy = \int_0^\infty 2\pi r e^{-r^2}dr = -\pi e^{-r^2}\Big|_0^\infty = \pi \Rightarrow$$

$$\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2.$$

Therefore, $\int_0^\infty e^{ix^2}dx = \frac{\sqrt{\pi}}{2}e^{i\pi/4} = \frac{\sqrt{\pi}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{\sqrt{\pi}}{2\sqrt{2}}(1+i)$

$$\Rightarrow \int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$ ✓