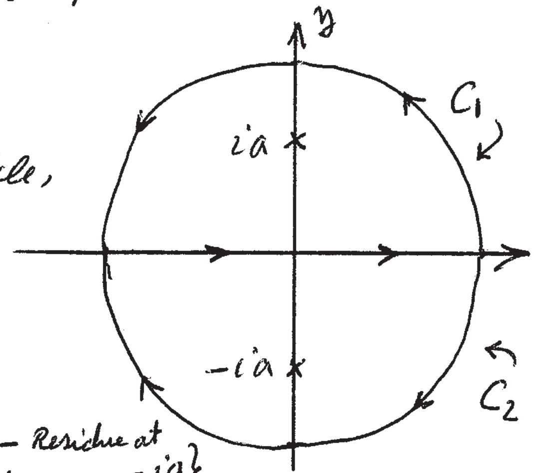
**Problem 13)**

a) $$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ix}}{(x+ia)(x-ia)}\,dx + \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{-ix}}{(x+ia)(x-ia)}\,dx$$

$$= \frac{1}{2}\oint_{C_1} \frac{e^{iz}}{(z+ia)(z-ia)}\,dz + \frac{1}{2}\oint_{C_2} \frac{e^{-iz}}{(z+ia)(z-ia)}\,dz$$




Jordan's lemma ensures that the first integral goes to zero on the upper semi-circle, while the second integral vanishes on the lower semi-circle. We thus have:

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx = \frac{1}{2}(2\pi i)\left\{\text{Residue at } ia - \text{Residue at } -ia\right\}$$

because $C_2$ is traversed clockwise.

$$= \pi i\left\{\frac{e^{i(ia)}}{ia+ia} - \frac{e^{-i(-ia)}}{-ia-ia}\right\} = \frac{\pi e^{-a}}{a}$$

If $k>0$, everything remains the same, but $e^{-a}$ becomes $e^{-ka}$.

If $k<0$, the first integral must be done on $C_2$, the second one on $C_1$. Then $e^{-a}$ becomes $e^{+ka}$. Therefore, the final result will be $\underbrace{\frac{\pi}{a}\exp(-|k|a)}$.

b) $$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}\,dx = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2}\,dx - \frac{1}{2i}\int_{-\infty}^{\infty} \frac{xe^{-ix}}{x^2+a^2}\,dx$$

$$= \frac{1}{2i}\oint_{C_1} \frac{ze^{iz}}{(z+ia)(z-ia)}\,dz - \frac{1}{2i}\oint_{C_2} \frac{ze^{-iz}}{(z+ia)(z-ia)}\,dz$$

$$= \frac{1}{2i}(2\pi i)\left\{\frac{iae^{i(ia)}}{ia+ia} + \frac{-iae^{-i(-ia)}}{-ia-ia}\right\} = \pi\left(\tfrac{1}{2}e^{-a} + \tfrac{1}{2}e^{-a}\right) = \pi e^{-a}$$ ✓

because $C_2$ is traversed clockwise.

Again, for $\sin(kx)$ the answer will be $\pm\pi e^{-|k|a}$, with $+$ for $k>0$, $-$ for $k<0$.