Problem 11) Both integrands, $\frac{1}{a+b\cos\theta}$ and $\frac{1}{a+b\sin\theta}$, are periodic, with a period of 2π. Since the integrals are taken over a full period (2π), a change of variable from θ to $\theta-\frac{\pi}{2}$ will convert the first integral into the second. (The limits of integration do not matter, as long as the integral is taken over a full period.)

Therefore, $\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}=\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}$.

By the same token, a change of variable from θ to $\theta-\pi$ shows that the integrals remain the same if $+b$ is replaced with $-b$ in the denominators. We could therefore ignore the case when $b<0$, and assume that $b>0$ in the following analysis. (The case of $b=0$ is trivial.)

In what follows, we will derive the integral $\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}$ for $a>b>0$.

On the unit-circle $z=e^{i\theta}$, $dz=ie^{i\theta}d\theta$, and $\cos\theta=\frac{1}{2}(z+\frac{1}{z})$. Thus

$$\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}=\oint_{\text{Unit Circle}}\frac{\overbrace{\left(\frac{dz}{iz}\right)}^{d\theta}}{a+\frac{b}{2}(z+\frac{1}{z})}=\frac{2}{i}\oint_{\text{Unit Circle}}\frac{dz}{2az+b(z^2+1)}$$

$$=-\frac{2i}{b}\oint_{\text{Unit Circle}}\frac{dz}{z^2+2(\frac{a}{b})z+1}=-\frac{2i}{b}\oint_{\text{Unit Circle}}\frac{dz}{(z-z_1)(z-z_2)}.$$

The roots of the denominator are $z_1, z_2=-(a/b)\pm\sqrt{(a/b)^2-1}$.

Since the product z_1z_2 is equal to 1, one of the roots must be inside the circle, and the other one outside. Since we have taken a and b as positive real numbers, it's clear that z_1 is inside.

the root with + sign

The residue at $z=z_1$ is therefore given by $\left.\frac{1}{z-z_2}\right|_{z=z_1} = \frac{1}{z_1-z_2} = \frac{1}{2\sqrt{(a/b)^2-1}}$.

The integral is thus given by:

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = -\left(\frac{2i}{b}\right)(2\pi i)\,\frac{1}{2\sqrt{(a/b)^2-1}} = \frac{2\pi}{\sqrt{a^2-b^2}}.$$

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