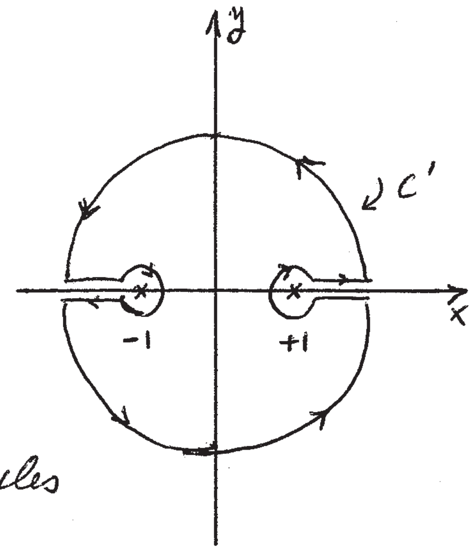
Poblem 10) $\oint_C \frac{dz}{z^2-1} = \oint_C \frac{dz}{(z-1)(z+1)}$



The integral over C' is zero because $\frac{1}{z^2-1}$ is analytic inside and over C'.
Thus the integral over circle C (radius $R=2$) is equal to the sum of the integrals over the two small circles, provided these circles are traversed counterclockwise.

For the small circle surrounding $z=1$, we let $z = 1+\epsilon e^{i\theta}$. Then

$$\oint_{\substack{\text{circle}\\ \text{at } z=1}} \frac{dz}{(z-1)(z+1)} = \int_{\theta=0}^{2\pi} \frac{i\cancel{\epsilon e^{i\theta}}\,d\theta}{\cancel{\epsilon e^{i\theta}}(2+\epsilon e^{i\theta})} \xrightarrow[\epsilon\to 0]{} \frac{2\pi i}{2} = \pi i. \checkmark$$

For the small circle surrounding $z=-1$, we let $z=-1+\epsilon e^{i\theta}$. Then

$$\oint_{\substack{\text{circle}\\ \text{at } z=-1}} \frac{dz}{(z-1)(z+1)} = \int_{\theta=0}^{2\pi} \frac{i\cancel{\epsilon e^{i\theta}}\,d\theta}{(-2+\epsilon e^{i\theta})\cancel{\epsilon e^{i\theta}}} \xrightarrow[\epsilon\to 0]{} \frac{2\pi i}{-2} = -\pi i. \checkmark$$

The sum of these two integrals is zero; therefore, $\oint_C \frac{dz}{z^2-1} = 0.$ ✓

Alternatively we could write: $\oint_C \frac{dz}{z^2-1} = \oint_C \left[\frac{1/2}{z-1} - \frac{1/2}{z+1}\right] dz$

$= \frac{1}{2}\oint_C \frac{dz}{z-1} - \frac{1}{2}\oint_C \frac{dz}{z+1}$. For each integral the residue at the simple pole is equal to 1. The answer, therefore, is

$\frac{1}{2}(2\pi i) - \frac{1}{2}(2\pi i) = 0$, as before.