Problem 7)

a) $u(x,y) = x^3 - 3xy^2$

Cauchy-Riemann: $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \Rightarrow \dfrac{\partial v}{\partial y} = 3x^2 - 3y^2 \Rightarrow v(x,y) = 3x^2y - y^3 + f(x) \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \Rightarrow \dfrac{\partial v}{\partial x} = +6xy \Rightarrow v(x,y) = 3x^2y + g(y) \end{cases}$

Comparing the above expressions for $v(x,y)$ it is obvious that $f(x)$ is a constant, while $g(y)$ is $-y^3$ plus a constant. Ignoring the constant we will have $v(x,y) = 3x^2y - y^3$. The function $f(z)$ is therefore given by:

$$f(z) = u(x,y) + i\,v(x,y) = (x^3 - 3xy^2) + i(3x^2y - y^3)$$
$$= (x+iy)^3 = z^3$$

b) $v(x,y) = e^{-y}\sin x$

Cauchy-Riemann: $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \Rightarrow \dfrac{\partial u}{\partial x} = -e^{-y}\sin x \Rightarrow u(x,y) = e^{-y}\cos(x) + f(y) \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \Rightarrow \dfrac{\partial u}{\partial y} = -e^{-y}\cos x \Rightarrow u(x,y) = e^{-y}\cos(x) + g(x) \end{cases}$

Comparing the above expressions for $u(x,y)$, it is clear that $f(x)$ and $g(y)$ can at best be equal to a constant. Ignoring this constant, we'll have $u(x,y) = e^{-y}\cos x$. The function $f(z)$ is thus given by:

$$f(z) = u(x,y) + i\,v(x,y) = e^{-y}\cos x + i e^{-y}\sin x = e^{-y}(\cos x + i\sin x)$$
$$= \exp(-y+ix) = \exp(iz). ✓$$