Problem 4) For $N = 0$, the polynomial reduces to $f(z) = a_0$, which is a constant function of z. The derivative of $f(z)$ is then zero everywhere in the complex z-plane, making $f(z)$ analytic everywhere. Suppose now that all N^{th} degree polynomials are differentiable throughout the complex plane. We must then prove that all polynomials of degree $N + 1$ are similarly differentiable across the complex plane. To this end, consider $f(z) = \sum_{n=0}^{N+1} a_n z^n$. The derivative of $f(z)$ at an arbitrary point z is given by

$$f'(z) = lim_{\Delta z \to 0} \left[\frac{f(z + \Delta z) - f(z)}{\Delta z}\right] = lim_{\Delta z \to 0} \left[\frac{\sum_{n=0}^{N+1} a_n (z + \Delta z)^n - \sum_{n=0}^{N+1} a_n z^n}{\Delta z}\right]$$

$$= a_{N+1}\, lim_{\Delta z \to 0} \left[\frac{(z + \Delta z)^{N+1} - z^{N+1}}{\Delta z}\right] + lim_{\Delta z \to 0} \left[\frac{\sum_{n=0}^{N} a_n (z + \Delta z)^n - \sum_{n=0}^{N} a_n z^n}{\Delta z}\right].$$

In the above expression, the first term is the constant a_{N+1} times the derivative of z^{N+1}, which, according to Problem 3, is given by $a_{N+1}(N + 1)z^N$. The second term is the derivative of an N^{th} degree polynomial, which, by assumption, exists. Consequently, the sum of the two terms comprising $f'(z)$ exists everywhere in the complex plane, proving that all polynomials of degree $N + 1$ are analytic across the entire complex plane.