Problem 2)

$$f(z)=\frac{1}{z^2+1}=\frac{1}{(x+\mathrm{i}y)^2+1}=\frac{1}{(x^2-y^2+1)+2\mathrm{i}xy}$$

$$=\frac{(x^2-y^2+1)-2\mathrm{i}xy}{(x^2-y^2+1)^2+4x^2y^2}=\frac{(x^2-y^2+1)-2\mathrm{i}xy}{(x^2+y^2)^2+2(x^2-y^2)+1}.$$

$$u(x,y)=\frac{x^2-y^2+1}{(x^2+y^2)^2+2(x^2-y^2)+1};\qquad v(x,y)=\frac{-2xy}{(x^2+y^2)^2+2(x^2-y^2)+1}.$$

$$\frac{\partial u}{\partial x}=\frac{2x[(x^2+y^2)^2+2(x^2-y^2)+1]-4x(x^2+y^2+1)(x^2-y^2+1)}{[(x^2+y^2)^2+2(x^2-y^2)+1]^2}$$

$$=-\frac{2x(x^4-3y^4-2x^2y^2+2x^2+2y^2+1)}{[(x^2+y^2)^2+2(x^2-y^2)+1]^2}.$$

$$\frac{\partial u}{\partial y}=\frac{-2y[(x^2+y^2)^2+2(x^2-y^2)+1]-4y(x^2+y^2-1)(x^2-y^2+1)}{[(x^2+y^2)^2+2(x^2-y^2)+1]^2}$$

$$=-\frac{2y(3x^4-y^4+2x^2y^2+2x^2+2y^2-1)}{[(x^2+y^2)^2+2(x^2-y^2)+1]^2}.$$

$$\frac{\partial v}{\partial x}=\frac{-2y[(x^2+y^2)^2+2(x^2-y^2)+1]+8x^2y(x^2+y^2+1)}{[(x^2+y^2)^2+2(x^2-y^2)+1]^2}$$

$$=\frac{2y(3x^4-y^4+2x^2y^2+2x^2+2y^2-1)}{[(x^2+y^2)^2+2(x^2-y^2)+1]^2}.$$

$$\frac{\partial v}{\partial y}=\frac{-2x[(x^2+y^2)^2+2(x^2-y^2)+1]+8xy^2(x^2+y^2-1)}{[(x^2+y^2)^2+2(x^2-y^2)+1]^2}$$

$$=-\frac{2x(x^4-3y^4-2x^2y^2+2x^2+2y^2+1)}{[(x^2+y^2)^2+2(x^2-y^2)+1]^2}.$$

Clearly, $\partial u/\partial x=\partial v/\partial y$ and $\partial u/\partial y=-\,\partial v/\partial x$. The only points where the function $f(z)$ is undefined are the roots of the denominator, namely, $z_{1,2}=\pm\mathrm{i}$. Aside from these two points, the function is analytic everywhere in the complex plane.