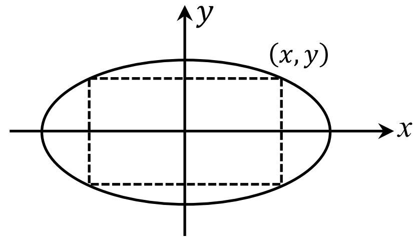
**Problem 15**) Pick a point $(x, y)$ on the ellipse. The area to be maximized is $f(x, y) = 4xy$, while the constraint is specified as $g(x, y) = (x/a)^2 + (y/b)^2 = 1$. We will have




$$F(x, y) = f(x, y) + \lambda g(x, y) = 4xy + \lambda(x/a)^2 + \lambda(y/b)^2.$$

Setting the partial derivatives of $F(x, y)$ with respect to $x$ and $y$ equal to zero, we find

$$\begin{cases} \dfrac{\partial F}{\partial x} = 4y + \dfrac{2\lambda x}{a^2} = 0 \\ \dfrac{\partial F}{\partial y} = 4x + \dfrac{2\lambda y}{b^2} = 0 \end{cases} \quad \rightarrow \quad \begin{pmatrix} \lambda/a^2 & 2 \\ 2 & \lambda/b^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The above equation will have a non-trivial solution only if the determinant of the $2 \times 2$ coefficient matrix vanishes, that is,

$$\begin{vmatrix} \lambda/a^2 & 2 \\ 2 & \lambda/b^2 \end{vmatrix} = \left(\frac{\lambda}{ab}\right)^2 - 4 = 0 \qquad \rightarrow \qquad \lambda = \pm 2ab.$$

With this value of $\lambda$, the solution of the pair of equations $\partial F/\partial x = 0,\ \partial F/\partial y = 0$ will be $y = \mp(b/a)x$. Putting this solution into the constraint equation now yields

$$g(x, y) = (x/a)^2 + (y/b)^2 = (x/a)^2 + (b/a)^2(x/b)^2 = 2(x/a)^2 = 1 \ \rightarrow \ x = \pm a/\sqrt{2}.$$

Consequently, $y = \pm b/\sqrt{2}$. The solutions we have arrived at specify the four corners of the desired rectangle. The area of the rectangle is then equal to $4xy = 2ab$. The ratio of the area of the maximal rectangle to that of the ellipse is thus $2ab/(\pi ab) = 2/\pi$.