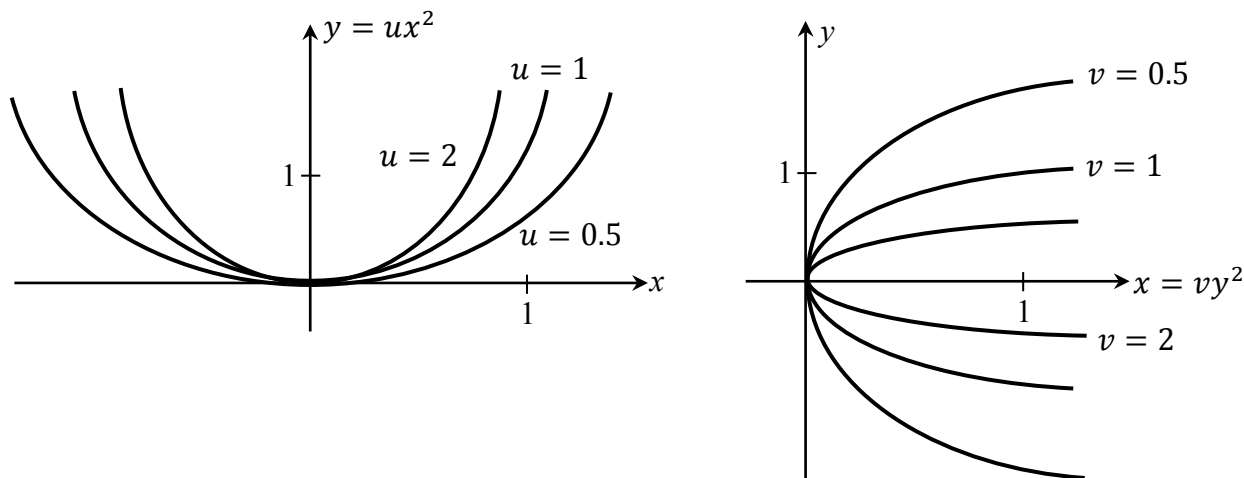


Problem 14) a) The following figures show several contours of constant u and constant v within the xy -plane. (Similar curves can be drawn for negative values of u and v as well.) Note that, within each quadrant of the xy -plane, the contours of constant u cover the entire quadrant (except for the y -axis), the contours of constant v also cover the entire quadrant (except for the x -axis), and, aside from the point at the origin, there exists only one point in any given quadrant where a constant- u contour crosses a constant- v contour. The crossing point identifies the unique Cartesian (x, y) coordinates associated with the curvilinear (u, v) coordinates.



$$\begin{aligned} \text{b)} \quad u = y/x^2 &\rightarrow \partial u/\partial x = -2y/x^3 & \text{and} & \quad \partial u/\partial y = 1/x^2, \\ v = x/y^2 &\rightarrow \partial v/\partial x = 1/y^2 & \text{and} & \quad \partial v/\partial y = -2x/y^3. \end{aligned}$$

$$\text{Jacobian: } \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial v/\partial x \\ \partial u/\partial y & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} -2y/x^3 & 1/y^2 \\ 1/x^2 & -2x/y^3 \end{vmatrix} = \frac{3}{x^2y^2}.$$

Similarly,

$$\begin{aligned} x = vy^2 = vu^2x^4 &\rightarrow x = u^{-2/3}v^{-1/3}, \\ y = ux^2 = uv^2y^4 &\rightarrow y = u^{-1/3}v^{-2/3}. \end{aligned}$$

$$\begin{aligned} \partial x/\partial u &= -2/3u^{-5/3}v^{-1/3} & \text{and} & \quad \partial x/\partial v = -1/3u^{-2/3}v^{-4/3}, \\ \partial y/\partial u &= -1/3u^{-4/3}v^{-2/3} & \text{and} & \quad \partial y/\partial v = -2/3u^{-1/3}v^{-5/3}. \end{aligned}$$

$$\text{Jacobian: } \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} -2/3u^{-5/3}v^{-1/3} & -1/3u^{-4/3}v^{-2/3} \\ -1/3u^{-2/3}v^{-4/3} & -2/3u^{-1/3}v^{-5/3} \end{vmatrix} = \frac{1}{3u^2v^2}.$$

Considering that $uv = 1/(xy)$, it is readily seen that $\partial(u, v)/\partial(x, y)$ and $\partial(x, y)/\partial(u, v)$ are inverses of each other.

c) In the xy -plane, the shaded area is determined as follows:

$$\text{Area} = \int_{x=0}^1 \left(\int_{y=x^2}^{\sqrt{x}} dy \right) dx = \int_0^1 (\sqrt{x} - x^2) dx = \left(\frac{2}{3} x^{3/2} - \frac{1}{3} x^3 \right)_{x=0}^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

The same area may be computed in the uv coordinate system by recognizing that both u and v range from 1.0 to ∞ , and that the relevant Jacobian is $|\partial(x, y)/\partial(u, v)| = 1/(3u^2v^2)$.

$$\text{Area} = \int_{u=1}^{\infty} \int_{v=1}^{\infty} 1 \cdot \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv = \frac{1}{3} \int_1^{\infty} \frac{du}{u^2} \int_1^{\infty} \frac{dv}{v^2} = \frac{1}{3} \left(-\frac{1}{u} \right)_{u=1}^{\infty} \left(-\frac{1}{v} \right)_{v=1}^{\infty} = \frac{1}{3}.$$
