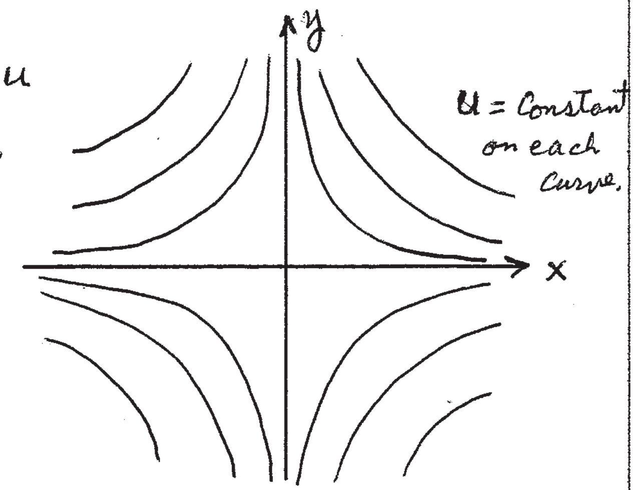
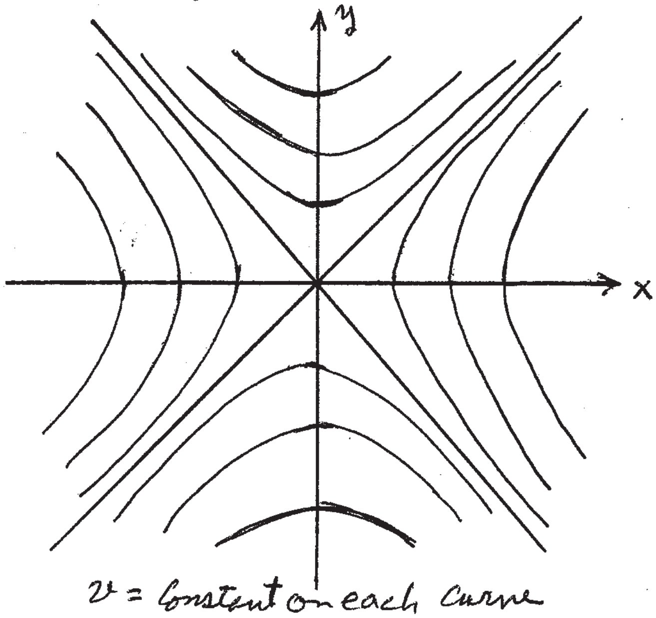
Problem 13) The plots of constant u are shown on the right. For $u>0$, the curves are in the 1st and 3rd quadrants. For $u<0$, the curves are in the 2nd and 4th quadrants. For $u=0$ the curves coincide with the x and y axes.





The plots of constant v are shown on the left. For $v>0$, the branches of constant v are those on the right- and left-hand sides. For $v<0$, the branches of constant v are those at the top and bottom. For $v=0$ the curves coincide with the straight lines $y=\pm x$.

$$xy=u \Rightarrow \begin{cases} \partial(xy)/\partial u = 1 \Rightarrow (\partial x/\partial u)y + x(\partial y/\partial u) = 1 & (1) \\ \partial(xy)/\partial v = 0 \Rightarrow (\partial x/\partial v)y + x(\partial y/\partial v) = 0 & (2) \end{cases}$$

$$x^2-y^2=v \Rightarrow \begin{cases} \partial(x^2-y^2)/\partial u = 0 \Rightarrow 2x(\partial x/\partial u) - 2y(\partial y/\partial u) = 0 & (3) \\ \partial(x^2-y^2)/\partial v = 1 \Rightarrow 2x(\partial x/\partial v) - 2y(\partial y/\partial v) = 1 & (4) \end{cases}$$

Equations (1) and (3) may now be solved for $\partial x/\partial u$ and $\partial y/\partial u$, as follows:

$$\begin{pmatrix} y & x \\ x & -y \end{pmatrix}\begin{pmatrix} \partial x/\partial u \\ \partial y/\partial u \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \partial x/\partial u \\ \partial y/\partial u \end{pmatrix} = \frac{1}{-y^2-x^2}\begin{pmatrix} -y & -x \\ -x & y \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow$$

$$\partial x/\partial u = \frac{y}{x^2+y^2}, \quad \partial y/\partial u = \frac{x}{x^2+y^2}.$$

Similarly, equations (2) and (4) may be solved for $\partial x/\partial v$ and $\partial y/\partial v$:

$$\begin{pmatrix} y & x \\ x & -y \end{pmatrix}\begin{pmatrix} \partial x/\partial v \\ \partial y/\partial v \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \Rightarrow \begin{pmatrix} \partial x/\partial v \\ \partial y/\partial v \end{pmatrix} = \frac{1}{-y^2-x^2}\begin{pmatrix} -y & -x \\ -x & y \end{pmatrix}\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$$

$$\Rightarrow \partial x/\partial v = \frac{x}{2(x^2+y^2)} \; ; \quad \partial y/\partial v = -\frac{y}{2(x^2+y^2)} .$$

The vectors $\vec{U}$ and $\vec{V}$ may now be written in terms of the partial derivatives as follows:

$$\vec{U} = \left(\frac{\partial x}{\partial u}\right)\hat{x} + \left(\frac{\partial y}{\partial u}\right)\hat{y} = \frac{y\hat{x} + x\hat{y}}{x^2+y^2}$$

$$\vec{V} = \left(\frac{\partial x}{\partial v}\right)\hat{x} + \left(\frac{\partial y}{\partial v}\right)\hat{y} = \frac{x\hat{x} - y\hat{y}}{2(x^2+y^2)}$$

It is now easy to see that $\vec{U}$ and $\vec{V}$ are orthogonal, because

$$\vec{U}\cdot\vec{V} = \frac{xy - xy}{2(x^2+y^2)^2} = 0 \text{ for all values of } x \text{ and } y.$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \\ \frac{x}{2(x^2+y^2)} & -\frac{y}{2(x^2+y^2)} \end{vmatrix} = -\frac{1}{2(x^2+y^2)} .$$