Problem 12) A small change from (x, y) to $(x + \Delta x, y + \Delta y)$ yields

$$\begin{cases} \Delta u = (\partial u/\partial x)\Delta x + (\partial u/\partial y)\Delta y \\ \Delta v = (\partial v/\partial x)\Delta x + (\partial v/\partial y)\Delta y \end{cases} \tag{1}$$

Consequently,

$$\begin{cases} 1 = \frac{\partial u}{\partial u} = \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial y}{\partial u}\right) \\ 0 = \frac{\partial v}{\partial u} = \left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial v}{\partial y}\right)\left(\frac{\partial y}{\partial u}\right) \end{cases} \quad \rightarrow \quad \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}\begin{pmatrix} \partial x/\partial u \\ \partial y/\partial u \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2}$$

Similarly

$$\begin{cases} 0 = \frac{\partial u}{\partial v} = \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial x}{\partial v}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial y}{\partial v}\right) \\ 1 = \frac{\partial v}{\partial v} = \left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial x}{\partial v}\right) + \left(\frac{\partial v}{\partial y}\right)\left(\frac{\partial y}{\partial v}\right) \end{cases} \quad \rightarrow \quad \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}\begin{pmatrix} \partial x/\partial v \\ \partial y/\partial v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3}$$

Combining Eqs.(2) and (3), we arrive at

$$\begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4}$$

The determinant of the product matrix in Eq.(4) being the product of the determinants of the individual matrices, we will have

$$\left[\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) - \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right)\right]\left[\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial y}{\partial v}\right) - \left(\frac{\partial x}{\partial v}\right)\left(\frac{\partial y}{\partial u}\right)\right] = 1 \quad \rightarrow \quad \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1. \tag{5}$$

The Jacobian for the forward transform is thus seen to be the inverse of the Jacobian of the inverse transform.