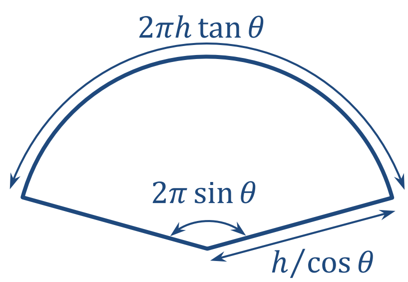
Problem 9)

a) The volume of a thin disk of radius $z\tan\theta$ and thickness dz is $dV = \pi(z\tan\theta)^2 dz$. Integrating from $z = 0$ to $z = h$ yields

$$V(h,\theta) = \int_0^h \pi z^2 \tan^2\theta\, dz = \pi\tan^2\theta \int_0^h z^2 dz = ⅓\pi h^3 \tan^2\theta. \qquad (1)$$

b) The perimeter of the base is $2\pi\rho = 2\pi h\tan\theta$, and the slanted height of the cone is $h/\cos\theta$. The surface area of the cone (i.e., the area of the flat sheet of paper out of which the cone is constructed) is thus given by

$$S(h,\theta) = ½(2\pi h\tan\theta)(h/\cos\theta) = \frac{\pi h^2 \sin\theta}{\cos^2\theta}. \qquad (2)$$



c) To minimize $S(h,\theta)$ subject to the constraint $V(h,\theta) = V_0$, we form the function $S + \lambda V$, where λ is the Lagrange multiplier, as follows:

$$S(h,\theta) + \lambda V(h,\theta) = \frac{\pi h^2 \sin\theta}{\cos^2\theta} + ⅓\lambda\pi h^3 \tan^2\theta. \qquad (3)$$

Setting to zero the partial derivatives of the above function with respect to the independent variables h and θ now yields

$$\frac{\partial}{\partial h}(S+\lambda V) = \frac{2\pi h\sin\theta}{\cos^2\theta} + \lambda\pi h^2\tan^2\theta = 0 \quad \rightarrow \quad \lambda h\sin\theta + 2 = 0. \qquad (4)$$

$$\frac{\partial}{\partial\theta}(S+\lambda V) = \pi h^2\left(\frac{\cos^3\theta + 2\cos\theta\sin^2\theta}{\cos^4\theta}\right) + ⅔\lambda\pi h^3\tan\theta\,(1+\tan^2\theta) = 0$$

$$\rightarrow \; 1 + \sin^2\theta + ⅔\lambda h\sin\theta = 0. \qquad (5)$$

Solving Eqs.(4) and (5) for h and θ, we find

$$\sin\theta = \frac{1}{\sqrt{3}} \qquad (\theta \cong 35.26°), \qquad (6a)$$

$$h = -2\sqrt{3}/\lambda. \qquad (6b)$$

Substituting h and θ into Eq.(1) in order to satisfy the constraint $V(h,\theta) = V_0$ now yields

$$V(h,\theta) = ⅓\pi h^3\tan^2\theta = -\frac{4\pi\sqrt{3}}{\lambda^3} = V_0 \quad\rightarrow\quad \lambda = -\left(\frac{4\pi\sqrt{3}}{V_0}\right)^{⅓} \rightarrow \quad h = \sqrt[3]{6V_0/\pi}. \qquad (7)$$

Finally, the minimum surface area is obtained by placing the optimum values of h and θ found in Eqs.(6a) and (7) into Eq.(2), that is,

$$S(h,\theta) = \frac{\pi h^2\sin\theta}{\cos^2\theta} = 3\left(\sqrt{3}\pi V_0^2/2\right)^{⅓}. \qquad (8)$$