

Problem 4) The function to be minimized is $S(r, h) = 2\pi r^2 + 2\pi r h$.

The constraint is $V(r, h) = \pi r^2 h = V_0$. We form the function $f(r, h) = 2\pi r^2 + 2\pi r h + \lambda \pi r^2 h$, where λ is the Lagrange multiplier. We'll have:

$$\begin{cases} \frac{\partial f}{\partial r} = 4\pi r + 2\pi h + 2\pi \lambda r h = 0 \Rightarrow 2r + (1 + \lambda r)h = 0 \Rightarrow h = -\frac{2r}{1 + \lambda r} \\ \frac{\partial f}{\partial h} = 2\pi r + \lambda \pi r^2 = 0 \Rightarrow r = 0, \text{ which is unacceptable, and } r = -\frac{2}{\lambda}. \end{cases}$$

Substituting $r = -\frac{2}{\lambda}$ in the expression for h yields: $h = -\frac{4}{\lambda}$.

Next, we use the constraint $\pi r^2 h = V_0$ to determine λ :

$$\pi \left(-\frac{2}{\lambda}\right)^2 \left(-\frac{4}{\lambda}\right) = V_0 \Rightarrow -\frac{16\pi}{\lambda^3} = V_0 \Rightarrow \lambda^3 = -\frac{16\pi}{V_0} \Rightarrow \lambda = -2\sqrt[3]{\frac{2\pi}{V_0}}.$$

$$\text{Therefore, } r = -\frac{2}{\lambda} = \sqrt[3]{\frac{V_0}{2\pi}} \quad \text{and} \quad h = -\frac{4}{\lambda} = 2\sqrt[3]{\frac{V_0}{2\pi}}.$$

It is readily verified that $\pi r^2 h = V_0$. The minimum surface area

$$\text{is found to be: } S(r, h) = 2\pi r^2 + 2\pi r h = 2\pi \left(\frac{V_0}{2\pi}\right)^{2/3} + 4\pi \left(\frac{V_0}{2\pi}\right)^{2/3} =$$

$$6\pi \left(\frac{V_0}{2\pi}\right)^{2/3} = 3(2\pi V_0^2)^{1/3}.$$

Digression: For a direct method of calculation of this problem,

$$\text{Let } h = \frac{V_0}{\pi r^2} \text{ into the expression for } S(r, h) \text{ to find: } S(r, h) = 2\pi r^2 + \frac{2V_0}{r}.$$

The minimum of S is then found by setting its derivative w.r.t. r equal to zero, that is, $4\pi r - \frac{2V_0}{r^2} = 0 \Rightarrow 4\pi r^3 = 2V_0 \Rightarrow r = \sqrt[3]{\frac{V_0}{2\pi}}$.

Note that $\frac{d^2 S}{dr^2} = 4\pi + \frac{4V_0}{r^3} \Big|_{r = \sqrt[3]{\frac{V_0}{2\pi}}} = 4\pi + 8\pi = 12\pi > 0$, and, therefore, the solution obtained is in fact a minimum, not a maximum.