Problem 17) Define $A=\binom{n}{0}+\binom{n}{3}+\binom{n}{6}+\cdots$, $B=\binom{n}{1}+\binom{n}{4}+\binom{n}{7}+\cdots$, $C=\binom{n}{2}+\binom{n}{5}+\binom{n}{8}+\cdots$, then invoke the binomial expansion $(1+x)^n=\sum_{m=0}^{n}\binom{n}{m}x^m$ to arrive at

$$[1+\exp(\mathrm{i}2\pi/3)]^n=\sum_{m=0}^{n}\binom{n}{m}\exp(\mathrm{i}2\pi m/3)=A+B\exp(\mathrm{i}2\pi/3)+C\exp(\mathrm{i}4\pi/3). \quad (1)$$

$$[1+\exp(\mathrm{i}4\pi/3)]^n=\sum_{m=0}^{n}\binom{n}{m}\exp(\mathrm{i}4\pi m/3)=A+B\exp(\mathrm{i}4\pi/3)+C\exp(\mathrm{i}2\pi/3). \quad (2)$$

$$(1+1)^n=\sum_{m=0}^{n}\binom{n}{m}=A+B+C. \quad (3)$$

On the left-hand side of Eq.(1), recalling that $\cos(\pi/3)=½$, we will have

$$[1+\exp(\mathrm{i}2\pi/3)]^n=e^{\mathrm{i}\pi n/3}(e^{-\mathrm{i}\pi/3}+e^{\mathrm{i}\pi/3})^n=e^{\mathrm{i}\pi n/3}[2\cos(\pi/3)]^n=e^{\mathrm{i}\pi n/3}. \quad (4)$$

Similarly, on the left-hand side of Eq.(2), recalling that $\sin(\pi/6)=½$, we will have

$$[1+\exp(\mathrm{i}4\pi/3)]^n=[1-\exp(\mathrm{i}\pi/3)]^n=e^{\mathrm{i}\pi n/6}(e^{-\mathrm{i}\pi/6}-e^{\mathrm{i}\pi/6})^n$$
$$=e^{\mathrm{i}\pi n/6}[-2\mathrm{i}\sin(\pi/6)]^n=(-\mathrm{i})^n e^{\mathrm{i}\pi n/6}=e^{-\mathrm{i}\pi n/2}e^{\mathrm{i}\pi n/6}=e^{-\mathrm{i}\pi n/3}. \quad (5)$$

Substitution for A into Eqs.(1) and (2) from Eq.(3), namely, $A=2^n-B-C$, now yields

$$B[1-\exp(\mathrm{i}2\pi/3)]+C[1-\exp(\mathrm{i}4\pi/3)]=2^n-e^{\mathrm{i}\pi n/3} \quad (6)$$

$$B[1-\exp(\mathrm{i}4\pi/3)]+C[1-\exp(\mathrm{i}2\pi/3)]=2^n-e^{-\mathrm{i}\pi n/3}. \quad (7)$$

Given the identities

$$1-\exp(\mathrm{i}2\pi/3)=1-\cos(2\pi/3)-\mathrm{i}\sin(2\pi/3)=1+½-\mathrm{i}\sqrt{3}/2=\sqrt{3}\exp(-\mathrm{i}\pi/6), \quad (8)$$

$$1-\exp(\mathrm{i}4\pi/3)=1-\cos(4\pi/3)-\mathrm{i}\sin(4\pi/3)=1+½+\mathrm{i}\sqrt{3}/2=\sqrt{3}\exp(\mathrm{i}\pi/6), \quad (9)$$

equations (6) and (7) may now be streamlined and written (in matrix notation) as follows:

$$\sqrt{3}\begin{pmatrix} e^{-\mathrm{i}\pi/6} & e^{\mathrm{i}\pi/6} \\ e^{\mathrm{i}\pi/6} & e^{-\mathrm{i}\pi/6} \end{pmatrix}\begin{pmatrix} B \\ C \end{pmatrix}=\begin{pmatrix} 2^n-e^{\mathrm{i}\pi n/3} \\ 2^n-e^{-\mathrm{i}\pi n/3} \end{pmatrix}. \quad (10)$$

Consequently,

$$\begin{pmatrix} B \\ C \end{pmatrix}=\frac{1}{\sqrt{3}}\begin{pmatrix} e^{-\mathrm{i}\pi/6} & e^{\mathrm{i}\pi/6} \\ e^{\mathrm{i}\pi/6} & e^{-\mathrm{i}\pi/6} \end{pmatrix}^{-1}\begin{pmatrix} 2^n-e^{\mathrm{i}\pi n/3} \\ 2^n-e^{-\mathrm{i}\pi n/3} \end{pmatrix}$$

$$=\frac{1}{\sqrt{3}(e^{-\mathrm{i}\pi/3}-e^{\mathrm{i}\pi/3})}\begin{pmatrix} e^{-\mathrm{i}\pi/6} & -e^{\mathrm{i}\pi/6} \\ -e^{\mathrm{i}\pi/6} & e^{-\mathrm{i}\pi/6} \end{pmatrix}\begin{pmatrix} 2^n-e^{\mathrm{i}\pi n/3} \\ 2^n-e^{-\mathrm{i}\pi n/3} \end{pmatrix}$$

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$$= \frac{1}{3}\begin{bmatrix} 2^n + 2\sin(\tfrac{1}{3}\pi n - \tfrac{1}{6}\pi) \\ 2^n - 2\sin(\tfrac{1}{3}\pi n + \tfrac{1}{6}\pi) \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2^n + 2\cos(\tfrac{1}{3}\pi n - \tfrac{1}{6}\pi - \tfrac{1}{2}\pi) \\ 2^n + 2\cos(\tfrac{1}{3}\pi n + \tfrac{1}{6}\pi + \tfrac{1}{2}\pi) \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2^n + 2\cos[(n-2)\pi/3] \\ 2^n + 2\cos[(n+2)\pi/3] \end{bmatrix}. \qquad (11)$$

Finally, the value of A is obtained as follows: $\cos a + \cos b = 2\cos[(a+b)/2]\cos[(a-b)/2]$

$$A = 2^n - (B + C) = 2^n - \tfrac{2}{3}\{2^n + \overbrace{\cos[(n-2)\pi/3] + \cos[(n+2)\pi/3]}\}$$

$$= \tfrac{1}{3}[2^n - 4\cos(n\pi/3)\cos(2\pi/3)] = \tfrac{1}{3}[2^n + 2\cos(n\pi/3)]. \qquad (12)$$