Problem 15) a) $f(x) = \sum_{n=1}^{\infty}(x^n/n) \quad \rightarrow \quad f'(x) = \sum_{n=1}^{\infty} x^{n-1} = 1/(1-x)$

$$\rightarrow \; f(x) = c + \int \frac{\mathrm{d}x}{1-x} = c - \ln(1-x).$$

To determine the integration constant c, note that $f(0) = \sum_{n=1}^{\infty}(x^n/n)|_{x=0} = 0$. Considering that $f(0) = c - \ln 1 = 0$, we conclude that $c = 0$ and that, therefore, $f(x) = -\ln(1-x)$.

b) $g(x) = \sum_{n=1}^{\infty}[x^{n+1}/n(n+1)] \quad \rightarrow \quad g'(x) = \sum_{n=1}^{\infty}(x^n/n) = f(x)$

$$\rightarrow \; g(x) = c + \int f(x)\mathrm{d}x = c - \int \ln(1-x)\,\mathrm{d}x = c + x + (1-x)\ln(1-x).$$

In the above derivation, we have guessed the anti-derivative of $\ln(1-x)$. The result is readily confirmed by differentiating $(1-x)\ln(1-x)$, which yields $-\ln(1-x) - 1$. The undesirable -1 is subsequently eliminated by the additional term x in the anti-derivative. To determine the integration constant c, note that $g(0) = 0$. Consequently, $c + 0 + \ln 1 = 0$, which yields $c = 0$. We thus arrive at $g(x) = x + (1-x)\ln(1-x)$.

c) $h(x) = \sum_{n=1}^{\infty} \frac{x^{n+2}}{n(n+1)(n+2)} \quad \rightarrow \quad h'(x) = \sum_{n=1}^{\infty}[x^{n+1}/n(n+1)] = g(x)$

$$\rightarrow \; h(x) = c + \int g(x)\mathrm{d}x = c + \int [x + (1-x)\ln(1-x)]\mathrm{d}x$$

$$= c + \tfrac{1}{2}x^2 - \tfrac{1}{2}(1-x)^2\ln(1-x) + \tfrac{1}{4}(1-x)^2$$

In the above derivation, we have guessed the anti-derivative of $(1-x)\ln(1-x)$. The result is confirmed by differentiating $-\tfrac{1}{2}(1-x)^2\ln(1-x)$, which yields $(1-x)\ln(1-x) + \tfrac{1}{2}(1-x)$. The undesirable $\tfrac{1}{2}(1-x)$ is subsequently eliminated by the additional term $\tfrac{1}{4}(1-x)^2$ in the anti-derivative. To determine the integration constant c, we note that $h(0) = 0$. Consequently, $h(0) = c + \tfrac{1}{4} = 0$, which yields $c = -\tfrac{1}{4}$. We thus have $h(x) = \frac{3}{4}x^2 - \frac{1}{2}x - \frac{1}{2}(1-x)^2\ln(1-x)$.