

Problem 12)

a) $\ln(x + 1)|_{x=0} = \ln 1 = 0,$

$$\frac{d}{dx} \ln(x + 1) = \frac{1}{x+1} \Big|_{x=0} = 1,$$

$$\frac{d^2}{dx^2} \ln(x + 1) = \frac{d}{dx} (x + 1)^{-1} = -(x + 1)^{-2} \Big|_{x=0} = -1,$$

$$\frac{d^3}{dx^3} \ln(x + 1) = -\frac{d}{dx} (x + 1)^{-2} = 2(x + 1)^{-3} \Big|_{x=0} = 2,$$

$$\frac{d^4}{dx^4} \ln(x + 1) = \frac{d}{dx} 2(x + 1)^{-3} = -3! (x + 1)^{-4} \Big|_{x=0} = -3!,$$

$$\frac{d^5}{dx^5} \ln(x + 1) = -3! \frac{d}{dx} (x + 1)^{-4} = 4! (x + 1)^{-5} \Big|_{x=0} = 4!.$$

Therefore, $\ln(x + 1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots = \sum_{n=1}^{\infty} [(-1)^{n+1}/n] x^n$. Substituting for $\ln(x + 1)$ in $f(x)$ now yields

$$f(x) = \frac{x}{\ln(x+1)} = \frac{x}{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots} = \frac{1}{1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \dots} = 1 / \sum_{n=0}^{\infty} [(-1)^n / (n + 1)] x^n.$$

It is now easy to see that $f(x = 0) = 1$ and $f(x = -1) = \frac{1}{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots} = \frac{1}{\infty} = 0$.

b) $f(x) = 1 / \sum_{n=0}^{\infty} [(-1)^n / (n + 1)] x^n = \sum_{m=0}^{\infty} a_m x^m$

$$\rightarrow \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [(-1)^n a_m / (n + 1)] x^{n+m} = 1$$

$$\rightarrow \sum_{k=0}^{\infty} \sum_{n=0}^k [(-1)^n a_{k-n} / (n + 1)] x^k = 1 \quad \leftarrow \boxed{\text{Defining } k = m + n}$$

Setting $k = 0$, the above equation yields the value of the first coefficient as $a_0 = 1$. For $k \geq 1$, the coefficient of x^k must be zero, that is, $\sum_{n=0}^k [(-1)^n a_{k-n} / (n + 1)] = 0$. We thus find

$$k = 1: \quad a_1 - \frac{1}{2}a_0 = 0 \quad \rightarrow \quad a_1 = \frac{1}{2}.$$

$$k = 2: \quad a_2 - \frac{1}{2}a_1 + \frac{1}{3}a_0 = 0 \quad \rightarrow \quad a_2 = \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}.$$

$$k = 3: \quad a_3 - \frac{1}{2}a_2 + \frac{1}{3}a_1 - \frac{1}{4}a_0 = 0 \quad \rightarrow \quad a_3 = -\frac{1}{24} - \frac{1}{6} + \frac{1}{4} = \frac{1}{24}.$$

$$k = 4: \quad a_4 - \frac{1}{2}a_3 + \frac{1}{3}a_2 - \frac{1}{4}a_1 + \frac{1}{5}a_0 = 0 \quad \rightarrow \quad a_4 = \frac{1}{48} + \frac{1}{36} + \frac{1}{8} - \frac{1}{5} = -\frac{19}{720}.$$

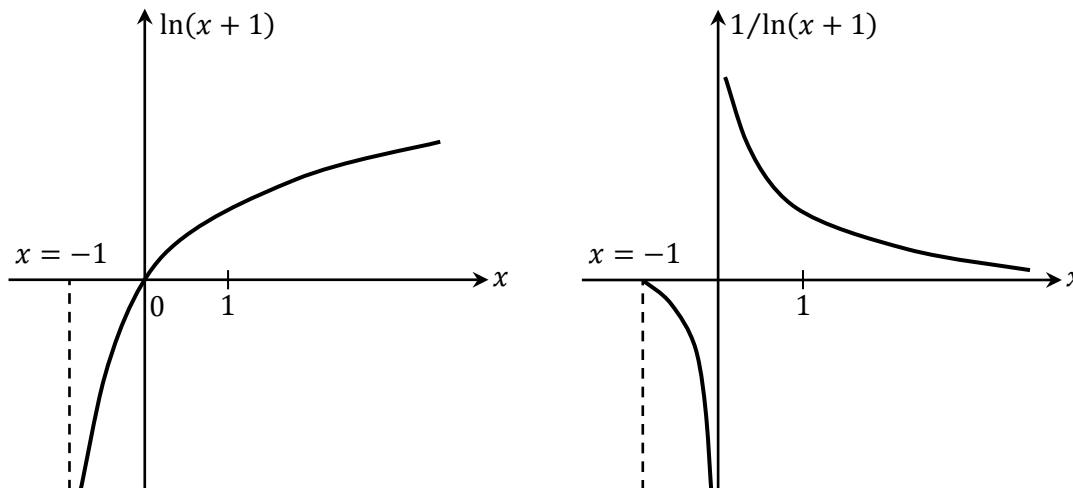
The Taylor series expansion of $f(x)$ is thus given by

$$f(x) = 1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{24} - \frac{19x^4}{720} + \dots$$

As a check on the validity of this expansion, note that the first five terms of the above Taylor series yield $f(-1) \cong 0.3486$, $f(-\frac{1}{2}) \cong 0.7223$, $f(0) = 1$, $f(\frac{1}{2}) \cong 1.2327$, $f(1) \cong 1.4319$,

while the actual values of the function are $f(-1) = 0$, $f(-\frac{1}{2}) = 0.7213 \dots$, $f(0) = 1$, $f(\frac{1}{2}) = 1.2331 \dots$, and $f(1) = 1.4426 \dots$.

c) The functions $\ln(x + 1)$ and $1/\ln(x + 1)$ are plotted below.



The function $f(x) = x/\ln(x + 1)$ is well defined over its domain $x \geq -1$, even at the ambiguous points $x = -1$ and $x = 0$; see part (a). The slope of the function at the ambiguous points is $f'(x = -1) = \infty$ and $f'(x = 0) = \frac{1}{2}$. A plot of $f(x)$ appears below.

