Problem 5) Since some of the terms are going to be subtracted from others, it is important to keep the elements of various sequences adjacent to each other. The terms in the denominator must therefore be $2n-1$, $2n$, and $2n+1$. We'll have:

$$\sum_{n=1}^{\infty} \frac{1}{n(2n-1)(2n+1)} = 2\sum_{n=1}^{\infty} \frac{1}{2n(2n-1)(2n+1)} = 2\left(\frac{A}{2n} + \frac{B}{2n-1} + \frac{C}{2n+1}\right)$$

$$= 2\,\frac{A(4n^2-1)+2Bn(2n+1)+2Cn(2n-1)}{2n(2n-1)(2n+1)} = \frac{4(A+B+C)n^2+2(B-C)n-A}{n(2n-1)(2n+1)} \Rightarrow$$

$$\begin{cases} A+B+C=0 \\ B-C=0 \\ -A=1 \end{cases} \Rightarrow A=-1,\ B=C=\tfrac{1}{2}$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(2n-1)(2n+1)} = -2\sum_{n=1}^{\infty} \frac{1}{2n} + \sum_{n=1}^{\infty} \frac{1}{2n-1} + \sum_{n=1}^{\infty} \frac{1}{2n+1}$

The last sum starts at $\frac{1}{3}$. Adding and subtracting 1 to this sum makes it similar to the second sum, that is, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1} = \sum_{n=1}^{\infty} \frac{1}{2n-1} - 1$.

We may then write:

$$\sum_{n=1}^{\infty} \frac{1}{n(2n-1)(2n+1)} = -2\sum_{n=1}^{\infty} \frac{1}{2n} + 2\sum_{n=1}^{\infty} \frac{1}{2n-1} - 1$$

$$= 2\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots\right) - 1.$$

We have shown in the class that $\ln 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$. (This was done by a Taylor series expansion of $\ln(1+x)$ around $x=0$, then evaluating the series at $x=1$.) The final result is: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(2n-1)(2n+1)} = 2\ln 2 - 1$