Problem 18) Use the Taylor series expansion of e^x, namely $e^x = \sum_{n=0}^{\infty}(x^n/n!)$, to arrive at

$$\int_0^1 e^{-x\ln x}\,\mathrm{d}x = \int_0^1 \sum_{n=0}^{\infty}[(-x\ln x)^n/n!]\mathrm{d}x = \sum_{n=0}^{\infty}[(-1)^n/n!]\int_0^1 x^n \ln^n x\,\mathrm{d}x. \qquad (1)$$

A change of variable from x to $y = -\ln x$ yields $x = e^{-y}$, $\mathrm{d}x = -e^{-y}\mathrm{d}y$, and, consequently,

$$\int_0^1 x^n \ln^n x\,\mathrm{d}x = -\int_{\infty}^0 e^{-ny}(-y)^n e^{-y}\mathrm{d}y = (-1)^n\int_0^{\infty} y^n e^{-(n+1)y}\mathrm{d}y = (-1)^n n!/(n+1)^{n+1}. \qquad (2)$$

The last integral in the preceding equation is obtained using the method of integration by parts (also listed in Gradshteyn & Ryzhik as **3.351**-3). Substitution into Eq.(1) finally yields

$$\int_0^1 e^{-x\ln x}\,\mathrm{d}x = \sum_{n=0}^{\infty}[(-1)^n/n!]\times[(-1)^n n!/(n+1)^{n+1}] = \sum_{n=0}^{\infty}(n+1)^{-(n+1)}. \qquad (3)$$