**Solution to Problem 16**) We begin by noting that

$$ax^2 + bx^{-2} = \left(\sqrt{a}x - \sqrt{b}x^{-1}\right)^2 + 2\sqrt{ab} = \sqrt{ab}\left(\sqrt[4]{a/b}\,x - \sqrt[4]{b/a}\,x^{-1}\right)^2 + 2\sqrt{ab}. \quad (1)$$

We then write

$$\int_0^\infty \exp(-ax^2 - bx^{-2})\,\mathrm{d}x = \sqrt[4]{b/a}\exp\left(-2\sqrt{ab}\right)\int_{-\infty}^{\infty}\exp\left[-\sqrt{ab}(e^y - e^{-y})^2\right]\exp(y)\,\mathrm{d}y$$

Change of variable: $\sqrt[4]{a/b}\,x = e^y$

Remove odd integrand

$$= \sqrt[4]{b/a}\exp\left(-2\sqrt{ab}\right)\int_{-\infty}^{\infty}\exp\left(-4\sqrt{ab}\sinh^2 y\right)(\cosh y + \sinh y)\mathrm{d}y$$

$$= \sqrt[4]{b/a}\exp\left(-2\sqrt{ab}\right)\int_{-\infty}^{\infty}\exp\left(-4\sqrt{ab}\sinh^2 y\right)\cosh(y)\,\mathrm{d}y$$

Change of variable: $x = 2\sqrt[4]{ab}\sinh(y)$

$$= \frac{\sqrt[4]{b/a}}{2\sqrt[4]{ab}}\exp\left(-2\sqrt{ab}\right)\int_{-\infty}^{\infty}\exp(-x^2)\,\mathrm{d}x = \sqrt{\pi/(4a)}\exp\left(-2\sqrt{ab}\right). \quad (2)$$

a) Changing the variable of integration from $\tau$ to $\theta$ such that $\cos^2\theta = \tau/T$, we will have

$$\int_{\tau=0}^{T}\exp\left(-\frac{a}{T-\tau} - \frac{b}{\tau}\right)\frac{\mathrm{d}\tau}{\sqrt{(T-\tau)\tau^3}} = \int_{\theta=0}^{\pi/2}\exp\left(-\frac{a/T}{1-\cos^2\theta} - \frac{b/T}{\cos^2\theta}\right)\frac{2T\sin\theta\cos\theta\,\mathrm{d}\theta}{T^2\sqrt{(1-\cos^2\theta)\cos^6\theta}}$$

$$= \frac{2}{T}\int_0^{\pi/2}\exp\left(-\frac{a/T}{\sin^2\theta} - \frac{b/T}{\cos^2\theta}\right)\frac{\mathrm{d}\theta}{\cos^2\theta}$$

$$= (2/T)\int_0^{\pi/2}\exp[-(a/T)(1+\cot^2\theta) - (b/T)(1+\tan^2\theta)](1+\tan^2\theta)\mathrm{d}\theta$$

Change of variable: $x = \tan\theta$

$$= (2/T)\exp[-(a+b)/T]\int_0^\infty \exp[-(a/T)x^{-2} - (b/T)x^2]\mathrm{d}x. \quad (3)$$

The resulting integral is similar to that given by Eq.(2). We thus have

$$\int_0^T \exp\left(-\frac{a}{T-\tau} - \frac{b}{\tau}\right)\frac{\mathrm{d}\tau}{\sqrt{(T-\tau)\tau^3}} = (2/T)\exp[-(a+b)/T]\sqrt{\pi T/(4b)}\exp\left(-2\sqrt{ab/T^2}\right)$$

$$= \sqrt{\pi/(bT)}\exp[-(\sqrt{a}+\sqrt{b})^2/T]. \quad (4)$$

b) Changing the variable of integration from $\tau$ to $\theta$ such that $\cos^2\theta = \tau/T$, we will have

$$\int_{\tau=0}^{T}\exp\left(-\frac{a}{T-\tau} - \frac{b}{\tau}\right)\frac{\mathrm{d}\tau}{[(T-\tau)\tau]^{3/2}} = \int_{\theta=0}^{\pi/2}\exp\left(-\frac{a/T}{1-\cos^2\theta} - \frac{b/T}{\cos^2\theta}\right)\frac{2T\sin\theta\cos\theta\,\mathrm{d}\theta}{T^3[(1-\cos^2\theta)\cos^2\theta]^{3/2}}$$

$$= \frac{2}{T^2}\int_0^{\pi/2}\exp\left(-\frac{a/T}{\sin^2\theta} - \frac{b/T}{\cos^2\theta}\right)\frac{\mathrm{d}\theta}{\sin^2\theta\cos^2\theta}$$

$$= \left(2/T^2\right)\int_0^{\pi/2}\exp[-(a/T)(1+\cot^2\theta) - (b/T)(1+\tan^2\theta)](1+\cot^2\theta)(1+\tan^2\theta)\mathrm{d}\theta$$

Change of variable: $x = \tan\theta$

$$= (2/T^2)\exp[-(a+b)/T]\int_0^\infty \exp[-(a/T)x^{-2} - (b/T)x^2](1+x^{-2})\mathrm{d}x. \quad (5)$$

The resulting integrals are similar to that in Eq.(2), albeit the second half of the integrand in Eq.(5) requiring a change of variable $y = x^{-1}$, which reverses the roles of $a$ and $b$. We will have

$$\int_0^T \exp\left(-\frac{a}{T-\tau} - \frac{b}{\tau}\right) \frac{\mathrm{d}\tau}{[(T-\tau)\tau]^{3/2}} = \left(2/T^2\right) \exp[-(a+b)/T] \left[\sqrt{\pi T/(4b)} + \sqrt{\pi T/(4a)}\right] \exp\left(-2\sqrt{ab/T^2}\right)$$

$$= \left[\sqrt{\pi/(aT^3)} + \sqrt{\pi/(bT^3)}\right] \exp[-(\sqrt{a}+\sqrt{b})^2/T]. \tag{6}$$

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