Solution to Problem 15) The goal here is to find an appropriate change of variable that would turn Euler's integral $\int_0^\infty e^{-t}t^n \mathrm{d}t$ into an integral that can be readily evaluated in the limit when $n \to \infty$. To this end, we must first examine the integrand $f(t) = e^{-t}t^n$.

a)
$$f'(t) = -e^{-t}t^n + ne^{-t}t^{n-1} = (n-t)e^{-t}t^{n-1} = 0 \quad \rightarrow \quad t_0 = n. \tag{1}$$

$$f''(t) = e^{-t}t^n - 2ne^{-t}t^{n-1} + n(n-1)e^{-t}t^{n-2}$$
$$= e^{-t}t^{n-2}[t^2 - 2nt + n(n-1)] = 0 \quad \rightarrow \quad t_{1,2} = n \pm \sqrt{n}. \tag{2}$$

The function $f(t)$ is thus seen to peak at $t_0 = n$, and to have inflection points on both sides of the peak at a distance of $\pm\sqrt{n}$. A plot of $f(t)$ versus t reveals that it more and more resembles the Gaussian function $\exp[-(x-x_0)^2/w_0^2]$ as n becomes larger. A change of variable for Euler's integral is thus suggested by this resemblance to the Gaussian function, that is, $x = (t-n)/\sqrt{n}$. We will have

$$n! = \int_0^\infty e^{-t}t^n \mathrm{d}t = \int_{-\sqrt{n}}^\infty e^{-(n+\sqrt{n}x)}(n+\sqrt{n}x)^n\sqrt{n}\mathrm{d}x$$
$$= \sqrt{n}(n/e)^n \int_{-\sqrt{n}}^\infty e^{-\sqrt{n}x}\left(1+\frac{x}{\sqrt{n}}\right)^n \mathrm{d}x. \tag{3}$$

Given the way the variable x has been defined, the range of values of x over which the above integrand is substantial must be centered around $x = 0$, with a width no greater than a few units on either side of $x = 0$. Consequently, for large n, one may treat $x/\sqrt{n}$ as a small entity. Recalling that $\ln(1+\varepsilon) = \varepsilon - \tfrac{1}{2}\varepsilon^2 + \tfrac{1}{3}\varepsilon^3 - \cdots$, we will have

$$\ln\left(1+\frac{x}{\sqrt{n}}\right)^n = n\ln\left(1+\frac{x}{\sqrt{n}}\right) = n\left(\frac{x}{\sqrt{n}} - \frac{x^2}{2n} + \frac{x^3}{3n\sqrt{n}} - \cdots\right) = \sqrt{n}x - \tfrac{1}{2}x^2 + \frac{x^3}{3\sqrt{n}} + \cdots. \tag{4}$$

Substitution into Eq.(3) now yields

$$n! = \sqrt{n}(n/e)^n \int_{-\sqrt{n}}^\infty e^{-\sqrt{n}x} e^{\sqrt{n}x - \frac{1}{2}x^2 + (x^3/3\sqrt{n}) + \cdots} \mathrm{d}x$$
$$= \sqrt{n}(n/e)^n \int_{-\sqrt{n}}^\infty e^{-\frac{1}{2}x^2 + (x^3/3\sqrt{n}) + \cdots} \mathrm{d}x. \tag{5}$$

In the limit of large n, the lower limit of the integral can be replaced with $-\infty$, and terms of order x^3 and higher that appear in the exponent of the integrand can be safely ignored. The integral in Eq.(5) then approaches $\int_{-\infty}^\infty \exp(-\tfrac{1}{2}x^2)\,\mathrm{d}x = \sqrt{2\pi}$, yielding the final (asymptotic) result as $n! \sim \sqrt{2\pi n}(n/e)^n$. This is consistent, of course, with Stirling's upper and lower bounds on $n!$, since $\sqrt{2\pi} \cong 2.506628$, which is greater than $e^{7/8} \cong 2.398875$ and smaller than $e \cong 2.718282$.
