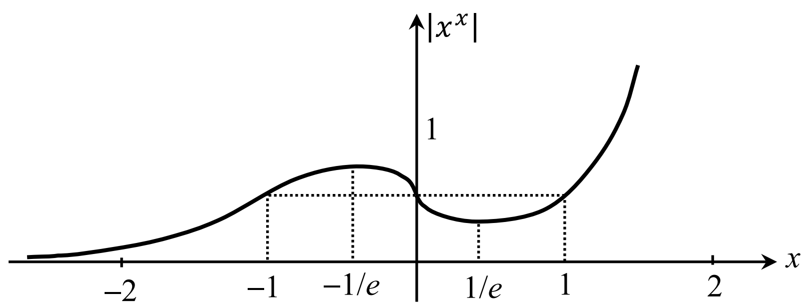
Problem 12) First consider the range of x where $x \geq 1$. Since $f(x) = x^x = \exp(x \ln x)$ is a monotonically increasing function of x when $x \geq 1$, it is easy to plot $f(x)$ by putting in a few values for x, say, $f(1) = 1$, $f(2) = 2^2 = 4$, $f(3) = 3^3 = 27$, and so on. For values of x between 0 and 1, we must find the derivative of $f(x)$ and see if there exist points at which the function may have a minimum or a maximum. We will have

$$f'(x) = (1 + \ln x)\exp(x \ln x) = (1 + \ln x)x^x = 0 \;\; \rightarrow \;\; \ln x = -1 \;\; \rightarrow \;\; x = 1/e \cong 0.368.$$

The value of the function at $x = 1/e$ is readily found to be $f(1/e) = e^{-(1/e)} = 0.692$. So, when x drops below 1.0, the function declines until $x = 1/e$, at which point it reaches its minimum, then begins to climb again as x falls below $1/e$ on its way to zero. Now, as x approaches zero from above, the limit of $x \ln x$ is 0, as can be verified by plugging in some small values for x, say, $x = 0.1$, 0.01, 0.001, which yield $x \ln x = -0.23$, -0.046, -0.0069. Consequently, $\lim_{x \to 0} x^x = 1$. Note that the slope of $f(x)$ approaches $-\infty$ as x goes to zero. On the negative half of the x-axis, where $\ln x = \ln|x| + \mathrm{i}\pi$, we have

$$x^x = \exp[x(\ln|x| + \mathrm{i}\pi)] = \exp(-|x|\ln|x|)\exp(\mathrm{i}\pi x)\,; \qquad (x < 0).$$

Therefore, the magnitude of x^x on the negative x-axis is seen to be the inverse of x^x on the positive half of the axis. In particular, the maximum of $|x^x|$ occurs at $x = -1/e$, where $|x^x| = e^{1/e} = 1.445$.



Shown above is a plot of $|x^x|$ over the entire x-axis. In addition, on the negative x-axis, the function is complex-valued, having a phase πx, which declines linearly as x goes from zero to $-\infty$. Note that, when $x = -1, -3, -5, \cdots$, the value of the phase-factor is $\exp(\mathrm{i}\pi x) = -1$, whereas for $x = -2, -4, -6, \cdots$, we have $\exp(\mathrm{i}\pi x) = 1$.