Problem 7)

a) $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i} = \dfrac{e^{ix-y} - e^{-ix+y}}{2i} = \dfrac{1}{2i}\left[e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right]$

$= \dfrac{1}{2i}\left[\cos x\,(e^{-y} - e^{+y}) + i\sin x\,(e^{-y} + e^{+y})\right] = \sin x \cosh y + i \cos x \sinh y.$

Therefore, $|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$

$= \sin^2 x \cosh^2 y + (1 - \sin^2 x)\sinh^2 y$

$= \sin^2 x\,(\underbrace{\cosh^2 y - \sinh^2 y}_{=1}) + \sinh^2 y = \sin^2 x + \sinh^2 y \geq \sin^2 x$

$\Rightarrow |\sin z| \geq |\sin x|$ ✓

b) We can use a method similar to that of part (a), but for the sake of diversity, we use a somewhat different method here.

$|\cos z|^2 = (\cos z)(\cos z)^* = \left(\dfrac{e^{iz} + e^{-iz}}{2}\right)\left(\dfrac{e^{iz} + e^{-iz}}{2}\right)^*$

$= \frac{1}{4}(e^{iz} + e^{-iz})(e^{-iz^*} + e^{+iz^*}) = \frac{1}{4}\left[e^{i(z-z^*)} + e^{i(z+z^*)} + e^{-i(z+z^*)}\right.$

$\left. + e^{-i(z-z^*)}\right] = \frac{1}{4}\left(e^{-2y} + e^{2ix} + e^{-2ix} + e^{+2y}\right)$

$= \frac{1}{4}\left\{(e^{y} - e^{-y})^2 + \cancel{2} + (e^{ix} + e^{-ix})^2 - \cancel{2}\right\} = \left(\dfrac{e^{y} - e^{-y}}{2}\right)^2 + \left(\dfrac{e^{ix} + e^{-ix}}{2}\right)^2$

$= \sinh^2 y + \cos^2 x \geq \cos^2 x \Rightarrow |\cos z| \geq |\cos x|$ ✓