Problem 3) In the method of integration by parts, the functions $f(x)$ and $g(x)$ satisfy the identity $\int_a^b f(x)g'(x)\mathrm{d}x = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)\mathrm{d}x$.

a) In the present problem, let $f(x) = \ln x$ and $g'(x) = x^n$. We will have

$$\int_0^1 x^n \ln(x)\,\mathrm{d}x = \overset{0-0}{\cancel{\frac{x^{n+1}\ln x}{n+1}\Big|_0^1}} - \frac{1}{n+1}\int_0^1 x^{n+1}x^{-1}\mathrm{d}x$$

$$= -\frac{1}{n+1}\int_0^1 x^n \mathrm{d}x = -\frac{1}{(n+1)^2}x^{n+1}\Big|_0^1 = -\frac{1}{(n+1)^2}.$$

b) Here $f(x) = \ln x$ and $g'(x) = 1/(1+x)^2$. We will have

$$\int_0^1 \frac{\ln x}{(1+x)^2}\mathrm{d}x = \lim_{\varepsilon\to 0}\left[-\frac{\ln x}{1+x}\Big|_\varepsilon^1 + \int_\varepsilon^1 \frac{\mathrm{d}x}{x(x+1)}\right] = \lim_{\varepsilon\to 0}\left[\frac{\ln\varepsilon}{1+\varepsilon} + \int_\varepsilon^1 \frac{\mathrm{d}x}{x} - \int_\varepsilon^1 \frac{\mathrm{d}x}{x+1}\right]$$

$$= \lim_{\varepsilon\to 0}\left[\frac{\ln\varepsilon}{1+\varepsilon} + \ln x|_\varepsilon^1 - \ln(x+1)|_\varepsilon^1\right]$$

$$= \lim_{\varepsilon\to 0}\left[\cancel{\frac{\ln\varepsilon}{1+\varepsilon}} - \cancel{\ln\varepsilon} - \ln 2 + \overset{0}{\cancel{\ln(1+\varepsilon)}}\right] = -\ln 2.$$

c) Change of variable: $y = \ln x$. Therefore, $\mathrm{d}y = (1/x)\mathrm{d}x$, or $\mathrm{d}x = \exp(y)\,\mathrm{d}y$. We find

$$\int_0^1 \frac{\ln x}{(1+x)^2}\mathrm{d}x = \int_{-\infty}^0 \frac{y\exp(y)}{[1+\exp(y)]^2}\mathrm{d}y = \int_{-\infty}^0 \frac{y}{[\exp(-y/2)+\exp(y/2)]^2}\mathrm{d}y$$

$$\boxed{\text{Change of variable: } x = y/2} \rightarrow = \int_{-\infty}^0 \frac{x}{\cosh^2(x)}\mathrm{d}x \quad \xrightarrow[\boxed{\text{Odd integrand}}]{} \quad \int_0^\infty \frac{x}{\cosh^2(x)}\mathrm{d}x = \ln 2.$$