Problem 15) a) It is easier to begin with finding the volume of a hemisphere of radius R, then multiply the result by 2 to arrive at the desired volume of the full sphere. Considering that the surface area of the n^{th} slice of the hemisphere is given by $\pi[1-(n/N)^2]R^2$, and that, according to chapter 1, problem 7, $\sum_{n=1}^{N} n^2 = N(N+1)(2N+1)/6$, we will have

$$V = lim_{N\to\infty} \sum_{n=0}^{N-1} \pi[1-(n/N)^2]R^2(R/N) = lim_{N\to\infty}(\pi R^3/N)\left[N - \left(\sum_{n=0}^{N-1} n^2\right)/N^2\right]$$

$$= \pi R^3[1 - lim_{N\to\infty} N(N-1)(2N-1)/(6N^3)]$$

$$= \pi R^3[1 - \tfrac{1}{6}\, lim_{N\to\infty}(1-N^{-1})\,(2-N^{-1})] = \tfrac{2}{3}\pi R^3.$$

The volume of the full sphere is thus given by $4\pi R^3/3$.

b) Let a sphere of radius R be enclosed within a (concentric) sphere of radius $R+\Delta R$. The volume ΔV of the thin region between the two spheres is readily seen to be

$$\Delta V = (4\pi/3)[(R+\Delta R)^3 - R^3] = 4\pi R^2 \Delta R[1 + (\Delta R/R) + \tfrac{1}{3}(\Delta R/R)^2].$$

In the limit when $\Delta R/R$ becomes exceedingly small, one can safely ignore the second and third terms on the right-hand side of the above equation to arrive at $\Delta V \cong 4\pi R^2 \Delta R$. The volume ΔV, however, must be nearly equal to the surface area S of the inner sphere multiplied by the width ΔR of the gap that separates the two spheres. In other words, $S\Delta R \cong \Delta V \cong 4\pi R^2 \Delta R$, which yields $S = 4\pi R^2$.