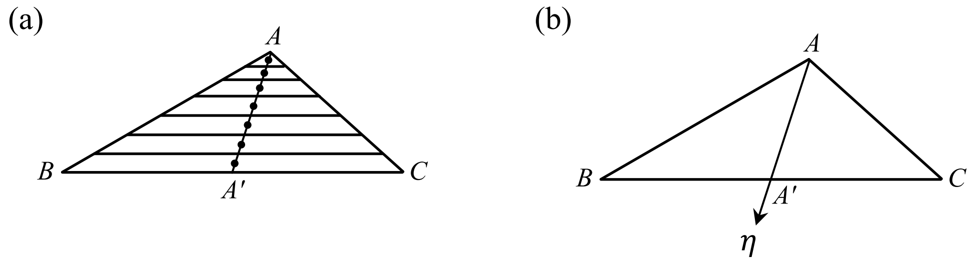
Problem 14) a) An easy way to find the center-of-mass of the triangular plate is to imagine that the plate is divided into numerous narrow strips parallel to one of the sides of the triangle, as shown in figure (a). Here the strips are parallel to BC, and the center-of-mass of each strip is located on the median AA', as indicated by the heavy dots placed at the center of each strip. Since the centers of mass associated with all the individual strips are lined up along the median AA', their collective center-of-mass, which is the center-of-mass of the entire plate, must be located somewhere along the median AA' as well. Now, there is nothing special about the base BC which was chosen to divide the plate into narrow strips. One could do the same with the base AC, in which case the center-of-mass would be located somewhere along the median BB'. Or, the strips could have been chosen parallel to AB, with the resulting center-of-mass falling somewhere along the median CC'. Considering that the plate's center-of-mass is unique, it is clear that the center-of-mass must be located at the point where the medians cross each other.



Alternatively, one may designate the median AA' as the η-axis, with origin of the axis at the vertex A, as shown in figure (b). The narrow strips parallel to BC will have mass proportional to η and also proportional to the width $\mathrm{d}\eta$ of the corresponding strip. Denoting the proportionality constant by the parameter α, the mass of each strip will be $\alpha\eta\mathrm{d}\eta$. Given that the centers of mass of all the strips are along the η-axis, the location of the plate's center-of-mass will be given by

$$\frac{\int_0^{\overline{AA'}} \eta(\alpha\eta)\mathrm{d}\eta}{\int_0^{\overline{AA'}} \alpha\eta\mathrm{d}\eta} = \frac{⅓\alpha(\overline{AA'})^3}{½\alpha(\overline{AA'})^2} = \frac{2}{3}\overline{AA'}.$$

The point located a distance of $⅔\overline{AA'}$ from the vertex A along the median AA' is the crossing point of the three medians of the triangle. Thus, the plate's center-of-mass is at the crossing point of its medians.

b) Let ρ be the mass density (per unit area) of the plate, and g the gravitational constant. Denoting by (x, y) the Cartesian coordinates of each point within the horizontal plate (balanced on the pencil tip), and by $\rho g\mathrm{d}x\mathrm{d}y\hat{\mathbf{z}}$ the gravitational force acting on the element $\mathrm{d}x\mathrm{d}y$ of the plate, the gravitational torque with respect to the center-of-mass (x_c, y_c) will be given by

$$\text{torque} = \iint_{\text{plate area}}[(x - x_c)\hat{\mathbf{x}} + (y - y_c)\hat{\mathbf{y}}] \times \rho g\mathrm{d}x\mathrm{d}y\hat{\mathbf{z}}$$

$$= \rho g[(\iint y\mathrm{d}x\mathrm{d}y - y_c \iint \mathrm{d}x\mathrm{d}y)\hat{\mathbf{x}} - (\iint x\mathrm{d}x\mathrm{d}y - x_c \iint \mathrm{d}x\mathrm{d}y)\hat{\mathbf{y}}].$$

Considering that $x_c = \iint_{\text{plate}} x\mathrm{d}x\mathrm{d}y / \iint_{\text{plate}} \mathrm{d}x\mathrm{d}y$ and $y_c = \iint_{\text{plate}} y\mathrm{d}x\mathrm{d}y / \iint_{\text{plate}} \mathrm{d}x\mathrm{d}y$, it is seen that, when the plate's center-of-mass is placed on the tip of the pencil, both components of the torque acting on the plate will vanish.