Problem 11)

a) Applying the theorem of Pythagoras to the diagram in Fig.(a), we write

$$d^2 = (R+h)^2 - R^2$$

$$\rightarrow \; d = \sqrt{2Rh + h^2} = \sqrt{2Rh}\,[1 + (h/2R)]^{½} = \sqrt{2Rh}\,\left[1 + \frac{h}{4R} - \frac{h^2}{32R^2} + \cdots\right]. \qquad (1)$$

Given the Earth's approximate radius $R \cong 6{,}378$ km, and a typical human height $h \cong 2$ m, we find $d \cong 5051$ meter.

b) In the triangle AB_2C, the angle A is $90° + \alpha$, the angle B_2 is $90° - (\theta + \alpha)$, the length of B_2C is R, and the length of AC is $R - \delta = R - d\tan\alpha$. Using the identity $\overline{AC}/\sin B_2 = \overline{B_2C}/\sin A$, we write

$$\frac{R - d\tan\alpha}{\sin[90° - (\theta+\alpha)]} = \frac{R}{\sin(90° + \alpha)} \quad \rightarrow \quad \cos(\theta + \alpha) = [1 - (d/R)\tan\alpha]\cos\alpha. \qquad (2)$$

Using small-angle approximations $\sin x \cong x$ and $\cos x = 1 - ½x^2$ for $x \ll 1$, and considering that $d \cong \sqrt{2Rh}$, we may simplify Eq.(2) as follows:

$$\cos(\theta+\alpha) \cong \cos\alpha - \sqrt{2h/R}\,\sin\alpha \quad \rightarrow \quad 1 - ½(\theta+\alpha)^2 \cong 1 - ½\alpha^2 - \alpha\sqrt{2h/R}$$

$$\rightarrow \quad \theta \cong \left(\alpha^2 + \alpha\sqrt{8h/R}\right)^{½} - \alpha. \qquad (3)$$

For typical values of $h \cong 2$ m, $R \cong 6{,}378$ km, and $\alpha \cong 0.0003$ rad, we will have $\theta \cong 0.00045$ rad. This means that the boats must be separated by a minimum distance of 2.9 km on the ocean's surface for the observer to be able to distinguish them apart.