Problem 22) The goal of this problem is to demonstrate that

$$\int_0^\pi \exp(p\cos x)\cos(p\sin x)\cos(nx)\,\mathrm{d}x = \int_0^\pi \exp(p\cos x)\sin(p\sin x)\sin(nx)\,\mathrm{d}x = \frac{\pi p^n}{2n!}.$$

Both integrands are even functions of x. Therefore, the value of each integral doubles if we extend the integration range to the interval $[-\pi,\pi]$. We then write

$$\int_{-\pi}^{\pi} \exp(p\cos x)\exp(\mathrm{i}p\sin x)\exp(\pm\mathrm{i}nx)\,\mathrm{d}x$$

$$= \int_{-\pi}^{\pi} \exp(pe^{\mathrm{i}x})\exp(\pm\mathrm{i}nx)\,\mathrm{d}x \overset{\text{Integration by parts}}{=} \cancel{\mp(\mathrm{i}/n)\exp(pe^{\mathrm{i}x})\exp(\pm\mathrm{i}nx)\big|_{-\pi}^{\pi}}$$

$$\pm(\mathrm{i}/n)\int_{-\pi}^{\pi} \mathrm{i}pe^{\mathrm{i}x}\exp(pe^{\mathrm{i}x})\exp(\pm\mathrm{i}nx)\,\mathrm{d}x$$

$$= \mp(p/n)\int_{-\pi}^{\pi} \exp(pe^{\mathrm{i}x})\exp[\pm\mathrm{i}(n\pm 1)x]\,\mathrm{d}x.$$

If we choose the upper sign, the process can be repeated indefinitely, each time the denominator n increasing by one unit, until the result goes to zero. This demonstrates that our two original integrals are equal to each other. In contrast, if we choose the lower sign, each repetition of the above procedure reduces the denominator n by one unit, until the second exponential in the integrand becomes equal to 1.0, at which point the coefficient preceding the final integral will be $p^n/n!$, and the remaining integral will be

$$\int_{-\pi}^{\pi} \exp(pe^{\mathrm{i}x})\,\mathrm{d}x = \int_{-\pi}^{\pi}\left[1 + pe^{\mathrm{i}x} + (p^2/2!)e^{2\mathrm{i}x} + (p^3/3!)e^{3\mathrm{i}x} + \cdots\right]\mathrm{d}x = 2\pi.$$

We thus find

$$\int_{-\pi}^{\pi} \exp(p\cos x)\exp(\mathrm{i}p\sin x)\exp(-\mathrm{i}nx)\,\mathrm{d}x = 2\pi p^n/n!.$$

The real part of the above integral is the sum of the two desired integrals, albeit evaluated over the $[-\pi,\pi]$ interval. Since the two integrals were shown to be equal to each other, we simply divide the final result by two, to arrive at the stated solution.