

**Problem 14)**

$$\text{a) } \int_1^{N+1} x^{k-1} dx = \sum_{n=1}^N \int_n^{n+1} x^{k-1} dx \quad \rightarrow \quad \frac{1}{k} [(N+1)^k - 1] = \frac{1}{k} \sum_{n=1}^N [(n+1)^k - n^k]$$

$$\begin{aligned} \rightarrow (N+1)^k - 1 &= \sum_{n=1}^N \left[ \sum_{m=0}^k \binom{k}{m} n^m - n^k \right] = \sum_{n=1}^N \sum_{m=0}^{k-1} \binom{k}{m} n^m = \sum_{m=0}^{k-1} \binom{k}{m} \sum_{n=1}^N n^m \\ &\rightarrow \sum_{m=0}^{k-1} \binom{k}{m} S_N^{(m)} = (N+1)^k - 1. \end{aligned}$$

$$\text{b) } k = 1: \quad S_N^{(0)} = N \quad (\text{that is, } 1^0 + 2^0 + 3^0 + \dots + N^0 = N).$$

$$k = 2: \quad S_N^{(0)} + \binom{2}{1} S_N^{(1)} = (N+1)^2 - 1 \quad \rightarrow \quad S_N^{(1)} = N(N+1)/2.$$

$$k = 3: \quad S_N^{(0)} + \binom{3}{1} S_N^{(1)} + \binom{3}{2} S_N^{(2)} = (N+1)^3 - 1$$

$$\rightarrow N + 3N(N+1)/2 + 3S_N^{(2)} = N^3 + 3N^2 + 3N$$

$$\rightarrow S_N^{(2)} = N(2N^2 + 3N + 1)/6 = N(N+1)(2N+1)/6.$$

$$k = 4: \quad S_N^{(0)} + \binom{4}{1} S_N^{(1)} + \binom{4}{2} S_N^{(2)} + \binom{4}{3} S_N^{(3)} = (N+1)^4 - 1$$

$$\rightarrow N + 2N(N+1) + N(N+1)(2N+1) + 4S_N^{(3)} = N^4 + 4N^3 + 6N^2 + 4N$$

$$\rightarrow S_N^{(3)} = [N(N+1)/2]^2.$$

Repeating the procedure for successive values of  $k$  enables one to obtain closed form expressions for  $S_N^{(m)}$  corresponding to larger values of  $m$ .

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