Problem 10) We use proof by contradiction, assuming at the outset that there exists a *smallest* integer, N, that can be decomposed into prime factors in two different ways. We then show that a smaller integer can be found that has the same property. Suppose the two decompositions of N are written as follows:

$$N = p_1^{m_1} p_2^{m_2} \cdots p_j^{m_j} = q_1^{n_1} q_2^{n_2} \cdots q_k^{n_k}. \tag{1}$$

Since, by assumption, N is the smallest integer that can be decomposed in two different ways, none of the prime factors p_i on the left-hand side of Eq.(1) cancel out any of the prime factors q_ℓ on the right-hand side. Without loss of generality, we assume that p_1 is the smallest prime number appearing in Eq.(1), then write $q_\ell = \mu_\ell p_1 + \nu_\ell$, where μ_ℓ and ν_ℓ are positive integers, with $\mu_\ell \geq 1$ and $1 \leq \nu_\ell < p_1$. The right-hand side of Eq.(1) may now be written as $\alpha p_1^\eta + \beta$, where α, β, and η are positive integers, with $\beta = \nu_1^{n_1} \nu_2^{n_2} \cdots \nu_k^{n_k}$.

Next, we subtract αp_1^η from both sides of Eq.(1). Since $\alpha \geq 1$ and $\eta \geq 1$, the left-hand side of the equation ends up being an integer smaller than N, with p_1 (or an integer power of p_1) as one of its prime factors. On the right-hand side, we will have $\beta = \nu_1^{n_1} \nu_2^{n_2} \cdots \nu_k^{n_k}$, which can be further decomposed into prime factors less than p_1 (because $\nu_1, \nu_2, \cdots, \nu_k$ are all smaller than p_1). We now have a number smaller than N, that is, $N - \alpha p_1^\eta$, that is factored out in two different ways: once with p_1 as a prime factor, and a second time with prime factors that are all smaller then p_1. This contradicts our starting assumption that N is the smallest integer that can be factored out in two different ways. The conclusion is that every integer can be decomposed into prime factors in only one way. This is the well-known fundamental theorem of arithmetic.