Problem 9) Take the two rationals to be $a = m/n$ and $b = p/q$. Their average value will be

$$c = \frac{a+b}{2} = \frac{1}{2}\left(\frac{m}{n} + \frac{p}{q}\right) = \frac{mq+np}{2nq},$$

which is the ratio of two integers and is, therefore, a rational. We have thus found the rational number c between a and b. (Other choices are also possible, of course, but the statement of the problem requires only one number between a and b, and we have picked what is perhaps the simplest solution.)

To find an irrational number between a and b, we use a numerical example to clarify the procedure. Suppose $a = 0.5673899732 \cdots$ and $b = 0.5673954092 \cdots$. Starting from the left, we skip the digits that are common between a and b (in this case 0.5673) until we reach the first digit at which a differs from b. If the difference between these two digits happens to be greater than unity, then we pick a digit between the two. In the present example, however, the two digits are 8 and 9. Since we cannot pick an integer between 8 and 9, we choose the smaller of the two (8 in this case), and append it to the preceding digits to form a new number $c = 0.56738$. From this point on, no matter what digits we append to c, the resulting number will be smaller than b. However, we still need to ensure that c will be greater than a. We thus continue to scan the digits of a and try to find something larger. The next two digits of a are 99, which are already as large as possible, but then the next digit is 7, which can be replaced by 8 or 9. Appending these digits to c, we will have $c = 0.56738998$. (**Note**: If it so happens that all the remaining digits of a are 9s, we must set them all to zero and increment by one the last digit of a that is below 9. In other words, if a in our example were $0.567389999999 \cdots$, it would have had to be replaced at the outset by $0.567390000 \cdots$, which is the same number.) Having thus found a finite number of digits for c in a way that c will continue to remain between a and b irrespective of any digits that one might add to it, we now proceed to append a random sequence to c. The random sequence must not terminate and it must not become periodic at any point. One way to ensure that the random sequence has these properties is to use the digits of a known irrational number (such as $\sqrt{2}$ or e or π) for the continuation of c. The resulting c is thus going to be an irrational number between a and b.

The situation is going to be essentially the same if a and b are irrationals. (With irrationals, of course, we will not encounter an infinite sequence of 9s at the end of a number.) Once a number c with a finite number of digits has been constructed such that $a < c < b$, we can stop the process and use c as the desired rational number. Appending a random sequence (non repeating and non-terminating) to the end of c will then yield the desired irrational number.