Problem 8) Consider a real-valued function F, defined on the complex plane $z = x + \mathrm{i}y$. Suppose also that both z and its conjugate z^* appear in the argument of the function, so that the function may be written as $F(z, z^*)$. To find an extremum (i.e., minimum, maximum, or saddle point) of the function, we must set $\partial F/\partial x$ and $\partial F/\partial y$ equal to zero. We thus have

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F}{\partial z^*}\frac{\partial z^*}{\partial x} = \frac{\partial F}{\partial z} + \frac{\partial F}{\partial z^*} = 0 \qquad \leftarrow \text{(because } \partial z/\partial x = 1 \text{ and } \partial z^*/\partial x = 1\text{)},$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F}{\partial z^*}\frac{\partial z^*}{\partial y} = \mathrm{i}\left(\frac{\partial F}{\partial z} - \frac{\partial F}{\partial z^*}\right) = 0 \qquad \leftarrow \text{(because } \partial z/\partial y = \mathrm{i} \text{ and } \partial z^*/\partial y = -\mathrm{i}\text{)}.$$

The above equations are satisfied if and only if $\partial F/\partial z = 0$ and $\partial F/\partial z^* = 0$. We now apply this argument to derive the Schwarz inequality. Let $F(\lambda, \lambda^*) = \int_a^b |f(x) + \lambda g(x)|^2 \mathrm{d}x$, that is,

$$F(\lambda, \lambda^*) = \int_a^b |f(x)|^2 \mathrm{d}x + \lambda \int_a^b f^*(x)g(x)\mathrm{d}x + \lambda^* \int_a^b f(x)g^*(x)\mathrm{d}x + \lambda\lambda^* \int_a^b |g(x)|^2 \mathrm{d}x.$$

We now have

$$\partial F(\lambda, \lambda^*)/\partial\lambda = \int_a^b f^*(x)g(x)\mathrm{d}x + \lambda^* \int_a^b |g(x)|^2 \mathrm{d}x = 0 \quad \rightarrow \quad \lambda^* = -\frac{\int_a^b f^*(x)g(x)\mathrm{d}x}{\int_a^b |g(x)|^2 \mathrm{d}x},$$

$$\partial F(\lambda, \lambda^*)/\partial\lambda^* = \int_a^b f(x)g^*(x)\mathrm{d}x + \lambda \int_a^b |g(x)|^2 \mathrm{d}x = 0 \quad \rightarrow \quad \lambda = -\frac{\int_a^b f(x)g^*(x)\mathrm{d}x}{\int_a^b |g(x)|^2 \mathrm{d}x}.$$

Considering that $\int_a^b f^*(x)g(x)\mathrm{d}x = \left[\int_a^b f(x)g^*(x)\mathrm{d}x\right]^*$, the above solutions for λ and λ^* are seen to be consistent. Placing these values of λ and λ^* into the expression for $F(\lambda, \lambda^*)$, we find

$$F(\lambda, \lambda^*) = \int_a^b |f(x)|^2 \mathrm{d}x - \frac{2\left[\int_a^b f(x)g^*(x)\mathrm{d}x\right]\left[\int_a^b f^*(x)g(x)\mathrm{d}x\right]}{\int_a^b |g(x)|^2 \mathrm{d}x} + \frac{\left[\int_a^b f(x)g^*(x)\mathrm{d}x\right]\left[\int_a^b f^*(x)g(x)\mathrm{d}x\right]}{\int_a^b |g(x)|^2 \mathrm{d}x}.$$

$$= \frac{\left[\int_a^b |f(x)|^2 \mathrm{d}x\right]\left[\int_a^b |g(x)|^2 \mathrm{d}x\right] - \left|\int_a^b f(x)g^*(x)\mathrm{d}x\right|^2}{\int_a^b |g(x)|^2 \mathrm{d}x}.$$

Since $F(\lambda, \lambda^*)$ is known to be ≥ 0, the above expression leads directly to the Schwarz inequality.