Problem 7) Let $S_1(N) = 1 + 2 + 3 + \cdots + N$. Add the first term to the last term; you will obtain $N + 1$. Add the second term to the one before last; you will obtain $N + 1$ again. If N is even, the above operation can be repeated $N/2$ times and the final result will be $S_1(N) = N(N + 1)/2$.

If N is odd, $N - 1$ will be even and the above procedure yields $S_1(N - 1) = (N - 1)N/2$. Subsequently, $S_1(N) = S_1(N - 1) + N = [(N - 1)N/2] + N = N(N + 1)/2$. In both cases, the answer is thus seen to be the same, namely, $S_1(N) = N(N + 1)/2$.

Next, we consider $S_2(N) = 1^2 + 2^2 + 3^2 + \cdots + N^2$. Assume $S_2(N) = aN^3 + bN^2 + cN + d$, with the a, b, c, d coefficients to be determined. We will have $S_2(1) = a + b + c + d = 1$. Also,

$$\begin{aligned} S_2(N) - S_2(N-1) &= aN^3 + bN^2 + cN + \cancel{d} - a(N-1)^3 - b(N-1)^2 - c(N-1) - \cancel{d} \\ &= a\cancel{N^3} + b\cancel{N^2} + c\cancel{N} - a(\cancel{N^3} - 3N^2 + 3N - 1) - b(\cancel{N^2} - 2N + 1) - c(\cancel{N} - 1) \\ &= 3aN^2 - 3aN + a + 2bN - b + c \\ &= 3aN^2 + (2b - 3a)N + (a - b + c) = N^2. \end{aligned}$$

Therefore,

$$a = ⅓; \qquad 2b - 3a = 0 \;\rightarrow\; b = ½; \qquad a - b + c = 0 \;\rightarrow\; c = ⅙.$$

We also found earlier that $a + b + c + d = 1$, which yields $d = 0$. Consequently,

$$S_2(N) = ⅓N^3 + ½N^2 + ⅙N = ⅙N(2N^2 + 3N + 1) = N(N + 1)(2N + 1)/6.$$